PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

Anthony Lasenby and Chris Doran

COURSE AIMS

- To introduce Geometric Algebra as a new mathematical technique to add to your existing base as a theoretician or experimentalist.
- To develop applications of this new technique in the fields of classical mechanics, engineering, relativistic physics and gravitation.

Our aim is to introduce these new techniques through their applications, rather than as purely formal mathematics. These applications will be diverse, emphasising the generality and portability of geometric algebra. This will help to promote a more inter-disciplinary view of science.

A full handout will accompany each lecture, and 3 question sheets will accompany the course. All material related to this course is available from

http://www.mrao.cam.ac.uk/~clifford/ptIIIcourse

or follow the link Cavendish → Research → Geometric Algebra → Lectures

A Quick Tour

In the following weeks we will

- Discover a new, powerful technique for handling rotations in arbitrary dimensions, and analyse the insights this brings to the mathematics of Lorentz transformations.
- Uncover the links between rotations, bivectors and the structure of the Lie groups which underpin much of modern physics.
- Learn how to extend the concept of a complex analytic function in 2-d (*i.e.* a function satisfying the Cauchy-Riemann equations) to arbitrary dimensions, and how this is applied in quantum theory and electromagnetism.
- Unite all four Maxwell equations into a single equation ($\nabla F = J$), and develop new techniques for solving it.
- Combine many of the preceeding ideas to construct a gauge theory of gravitation in (flat) Minkowski spacetime, which is still consistent with General Relativity.
- Use our new understanding of gravitation to quickly reach advanced applications such as black holes and cosmic strings.

SOME HISTORY

A central problem being tackled in the first part of the 19th Century was how best to represent 3-d rotations.

1844

Hamilton introduces his quaternions, which generalize complex numbers. But confusion persists over the status of vectors in his algebra — do (i, j, k) constitute the components of a *vector*?

1844

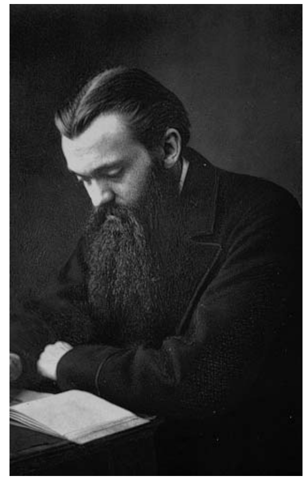
In a separate development, Grassmann. introductes of the exterior product. (See later this lecture.) Largely ignored in his lifetime, his work later gave rise to differential forms and Grassmann (anticommuting) variables (used in supersymmetry and superstring theory)

1878

Clifford invents Geometric Algebra by uniting the dot product and exterior products into a single geometric product. This is invertible, so an equation such as $ab = C$ has the solution $b = a^{-1}C$. This is not possible with the separate dot or exterior products.

Clifford could relate his product to the quaternions, and his system should have gone on to dominate mathematical physics. But . . .

- Clifford died young, at the age of just 34
- Gibbs introduced his vector calculus, which rapidly became very popular, and eclipsed Clifford and Grassmann's work.

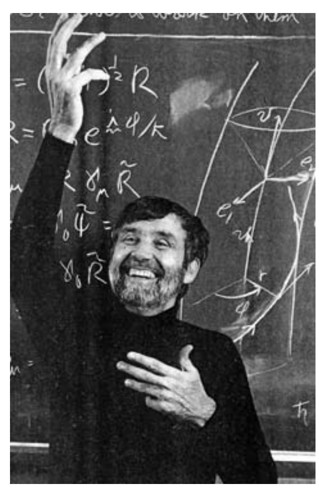


1920's

Clifford algebra resurfaces in the theory of quantum spin. In particular the algebra of the Pauli and Dirac matrices became indispensable in quantum theory. But these were treated just as algebras — the geometrical meaning was lost.

1966

David Hestenes recovers the geometrical meaning (in 3-d and 4-d respectively) underlying the Pauli and Dirac algebras. Publishes his results in the book Spacetime Algebra. Hestenes goes on to produce a fully developed geometric calculus.



In 1984, Hestenes and Sobczyk publish

Clifford Algebra to Geometric Calculus

This book describes a unified language for much for mathematics, physics and engineering. This was followed in 1986 by the (much easier!)

New Foundations for Classical Mechanics

1990's

Hestenes' ideas have been slow to catch on, but in Cambridge we now routinely apply geometric algebra to topics as diverse as

- black holes and cosmology (Astrophysics, Cavendish)
- quantum tunnelling and quantum field theory (Astrophysics, Cavendish)
- beam dynamics and buckling (Structures Group, CUED)
- computer vision (Signal Processing Group, CUED)

Exactly the same algebraic system is used throughout.

PART 1

GEOMETRIC ALGEBRA AND CLASSICAL MECHANICS

LECTURE 1

In this lecture we will introduce the basic ideas behind the mathematics of geometric algebra (abbreviated to GA). We will then focus on simple applications in 2-d. A full formal introduction will be delayed until Lecture 3

- Multiplying Vectors - The inner and cross products
- The Exterior Product - Encoding the geometry of planes and higher dimensional objects
- The Geometric Product - Axioms and properties
- The Geometric Algebra of 2-dimensional space
- Complex numbers rediscovered. The algebra of rotations has a particularly simple expression in 2-d, and leads to the identification of complex numbers with GA.
- Regularising Keplerian orbits. The GA treatment of rotations provides an alternative set of variables for describing elliptical orbits, which turn out to have many advantages.

MULTIPLYING VECTORS

In your mathematical training so far, you will have met two products for vectors:

1. The Inner Product

The inner, or dot product, is usually written in the form $a \cdot b$. (Note that we do not use bold for vectors any more.) In Euclidean space the inner product is positive definite,

$$a^2 = a \cdot a > 0 \qquad \forall a \neq 0$$

From this we recover Schwarz inequality

$$\begin{aligned} (a + \lambda b)^2 &> 0 \qquad \forall \lambda \\ \Rightarrow a^2 + 2\lambda a \cdot b + \lambda^2 b^2 &> 0 \qquad \forall \lambda \\ \Rightarrow (a \cdot b)^2 &\leq a^2 b^2 \end{aligned}$$

We use this to define the cosine of the angle between a and b via

$$a \cdot b = |a||b| \cos(\theta)$$

In non-Euclidean spaces, such as Minkowski space, we cannot do this. But we can still introduce an orthogonal frame and compute the dot product as *eg*. $a_\mu b^\mu$ or $\eta_{\mu\nu} a^\mu b^\nu$ where $\eta_{\mu\nu}$ is the metric tensor

2. The Cross Product

This only exists in 3-d space and is defined such that $a \times b$ is perpendicular to the plane defined by a and b, with magnitude $|a||b|\sin(\theta)$ and such that a, b and $a \times b$ form a right-handed set. This is sufficient to define the cross product uniquely. On introducing a right-handed orthonormal frame $\{e_i\}$ we can recover the usual definition in terms of components. We have

$$e_1 \times e_2 = e_3 \qquad \text{etc.}$$

Or, in more general index notation

$$e_i \times e_j = \epsilon_{ijk} e_k$$

If we now expand the vectors in terms of components, $a = a_i e_i$ and $b = b_i e_i$, we find

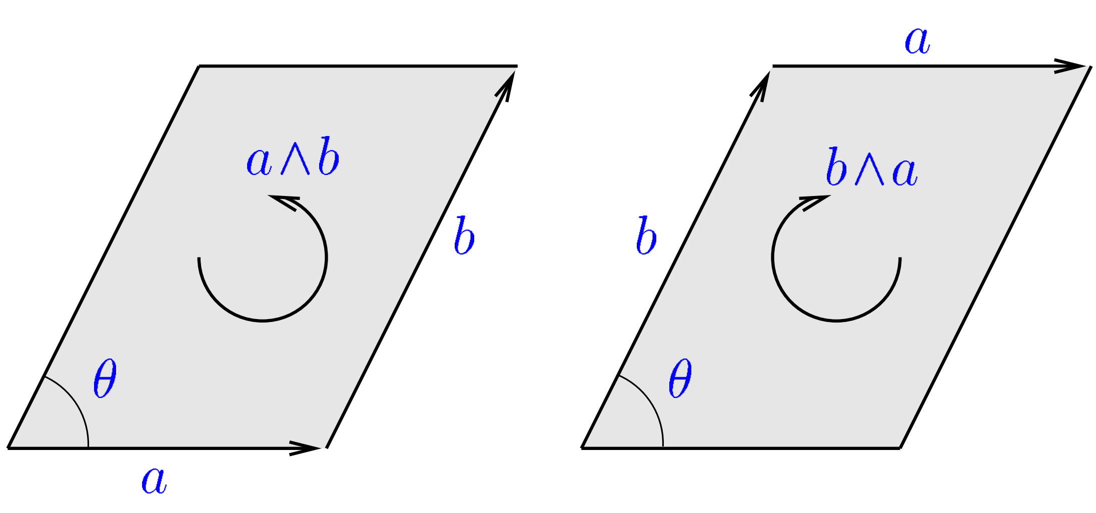
$$\begin{aligned} a \times b &= (a_i e_i) \times (b_j e_j) \\ &= a_i b_j (e_i \times e_j) \\ &= (\epsilon_{ijk} a_i b_j) e_k \end{aligned}$$

So the geometric definition recovers the algebraic one. One aim of GA is to extend this idea and avoid introducing frames as much as possible.

THE EXTERIOR PRODUCT

The cross product has one major failing - it only exists in 3 dimensions. In 2-d there is nowhere else to go, whereas in 4-d the concept of a vector orthogonal to a pair of vectors is not unique. To see this, consider 4 orthonormal vectors $e_1 \ldots e_4$. If we take the pair e_1 and e_2 and attempt to find a vector perpendicular to both of these, we see that any combination of e_3 and e_4 will do.

What we need is a means of encoding a plane geometrically, without relying on the notion of a vector perpendicular to it. We define the outer or wedge product $a \wedge b$ to be the directed area swept out by a and b. The plane has area $|a||b|\sin(\theta)$, which is defined to be the magnitude of $a \wedge b$.



The outer product of two vectors defines an oriented plane. This plane can be thought of as the parallelogram obtained by sweeping one vector along the other. Changing the order of the vectors reverses the orientation of the plane.

The result of the wedge product is neither a scalar nor a vector. It is a bivector — a new mathematical entity encoding the notion of a plane.

Properties

1. The outer product of two vectors is antisymmetric,

$$a \wedge b = -b \wedge a$$

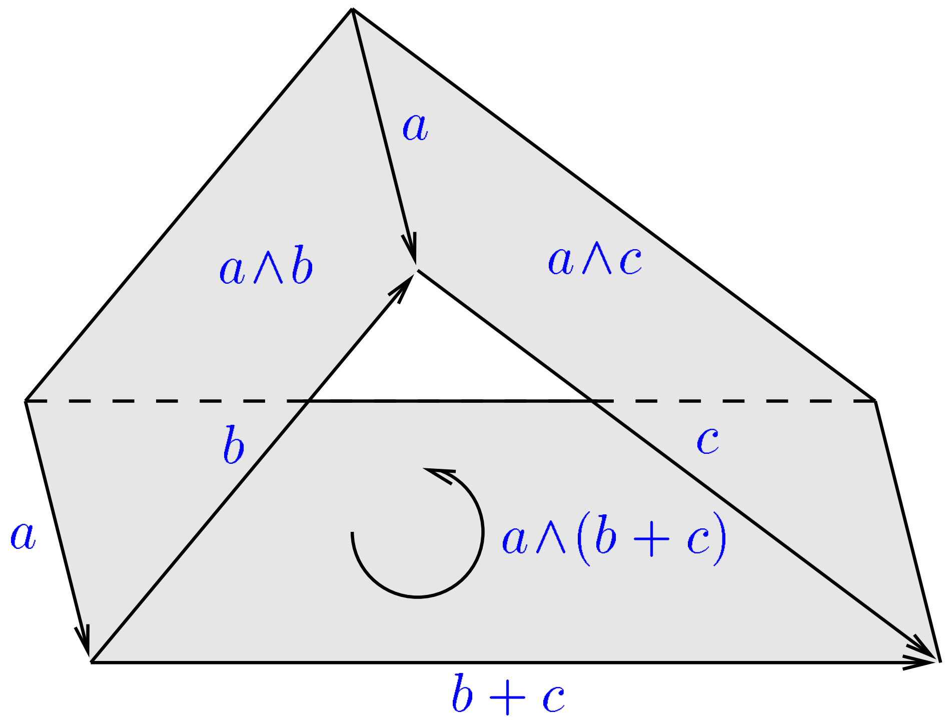
This follows from the geometric definition.

2. Bivectors form a linear space, the same way that vectors do. In 3-d the addition of bivectors is easy to visualise (see picture on next slide). In higher dimensions this addition is not always so easy to visualise, because two planes need not share a common line. This can have some interesting consequences.

3. The outer product is distributive

$$a \wedge (b + c) = a \wedge b + a \wedge c$$

This helps to visualise the addition of bivectors.



Note that if $a' = a + \lambda b$, we still have $a'\wedge b = a\wedge b$. There is no unique dependence on a and b. It is sometimes better to replace the directed parallelogram with a directed circle.

In 3-d the space of bivectors is three dimensional. An arbitrary bivector can be decomposed in terms of an orthonormal frame of bivectors.

$$
\begin{aligned}
a\wedge b &= (a_i e_i)\wedge(b_j e_j) \\
&= (a_2 b_3 - b_3 a_2) e_2\wedge e_3 + (a_3 b_1 - a_1 b_3) e_3\wedge e_1 \\
&\quad +(a_1 b_2 - a_2 b_1) e_1\wedge e_2
\end{aligned}
$$

The components in this frame are therefore those of the cross product. In general, the components of $a\wedge b$ are $a_{[i} b_{j]}$.

THE GEOMETRIC PRODUCT

So far we have a symmetric inner product and an antisymmetric outer product. Clifford's great idea was to introduce a new product which combines the two. This is the geometric product, written simply as ab, and satisfying

$$ab = a\cdot b + a\wedge b$$

The right-hand side is a sum of two distinct obejcts - a scalar and a bivector. This looks strange, and goes against much of what you might already have been taught. The easiest way to think of the right-hand side is like a complex number, with real and imaginary parts. These are carried round in a single entity, which provides for many mathematical simplifications.

From the symmetry/antisymmetry of the terms on the right-hand side, we see that

$$ba = b\cdot a + b\wedge a = a\cdot b - a\wedge b$$

It follows that

$$a\cdot b = \tfrac{1}{2}(ab + ba) \qquad a\wedge b = \tfrac{1}{2}(ab - ba)$$

We can thus define the other products in terms of the geometric product. This forms the starting point for an axiomatic development (Lecture 3). For the time being we will simply state some properties of the product.

1. General elements of a Geometric Algebra are called multivectors and these form a linear space - scalars can be added to bivectors, and vectors, *etc.*

2. The geometric product is associative

$$A(BC) = (AB)C = ABC$$

3. The geometric product is distributive

$$A(B + C) = AB + AC$$

   (Note that nothing is assumed about the commutation properties of the geometric product. Matrix multiplication is a good picture to keep in mind.)

4. The square of any vector is a scalar.

The final axiom is sufficient to prove that the inner product of two vectors is a scalar. Consider the expansion

$$\begin{aligned} (a+b)^2 &= (a+b)(a+b) \\ &= a^2 + b^2 + ab + ba \end{aligned}$$

It follows that

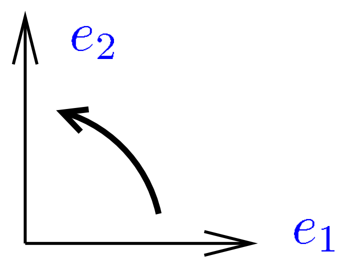
$$ab + ba = (a+b)^2 - a^2 - b^2$$

which is therefore a scalar.

GEOMETRIC ALGEBRA IN 2-D

The easiest way to understand the geometric product is by example, so consider a 2-d space (a plane) spanned by 2 orthonormal vectors e_1, e_2. These satisfy

$$e_1{}^2 = e_2{}^2 = 1$$
$$e_1 \cdot e_2 = 0$$



The final entity present in the 2-d algebra is the bivector $e_1 \wedge e_2$. This is the highest grade element in the algebra, which is often called the pseudoscalar, though directed volume element is a more accurate description. This is defined to be right-handed.

The full algebra ($\mathcal{G}_2$) is therefore spanned by

1	$\{e_1, e_2\}$	$e_1 \wedge e_2$
1 scalar	2 vectors	1 bivector

To study the properties of the bivector $e_1 \wedge e_2$ we first note that

$$e_1 e_2 = e_1 \cdot e_2 + e_1 \wedge e_2 = e_1 \wedge e_2$$

That is, for orthogonal vectors the geometric product is a pure bivector. Also note that

$$e_2 e_1 = e_2 \wedge e_1 = -e_1 \wedge e_2$$

from the antisymmetry of the exterior product. Another way of saying this is that in GA orthogonal vectors anticommute.

We can now form products when $e_1 e_2$ multiplies vectors from the left and the right. First from the left,

$$(e_1 \wedge e_2) e_1 = (-e_2 e_1) e_1 = -e_2 e_1 e_1 = -e_2$$
$$(e_1 \wedge e_2) e_2 = (e_1 e_2) e_2 = e_1 e_2 e_2 = e_1$$

We see that left multiplication by the bivector rotates vectors 90^{o} clockwise (*i.e.* in a negative orientation). Similarly, acting from the right

$$e_1(e_1 e_2) = e_2 \qquad e_2(e_1 e_2) = -e_1$$

So right multiplication rotates 90^{o} anticlockwise.

The final product in the algebra to consider is the square of the bivector $e_1 \wedge e_2$

$$(e_1 \wedge e_2)^2 = e_1 e_2 e_1 e_2 = -e_1 e_1 e_2 e_2 = -1$$

From purely geometric considerations, we have discovered a quantity which squares to -1. This fits with the fact that 2 successive left (or right) multiplications of a vector by $e_1 e_2$ rotates the vector through 180^{o}, which is equivalent to multiplying by -1.

Multiplying Multivectors

Suppose that we have two completely arbitrary elements of the $\mathcal{G}_2$ algebra, A and B. We can decompose these in terms of our $\{e_1, e_2\}$ frame as follows:

$$\begin{aligned} A &= a_0 + a_1 e_1 + a_2 e_2 + a_3 e_1 \wedge e_2 \\ B &= b_0 + b_1 e_1 + b_2 e_2 + b_3 e_1 \wedge e_2 \end{aligned}$$

The product of these two elements can be written

$$AB = p_0 + p_1 e_1 + p_2 e_2 + p_3 e_1 \wedge e_2$$

We find that

$$p_0 = a_0 b_0 + a_1 b_1 + a_2 b_2 - a_3 b_3$$

with similar formulae for p_1, p_2 and p_3. This multiplication law is easy to represent as part of a computer language (we often use Maple). The basis vectors can also be represented with matrices, though these can hide the geometry of the algebra.

If we introduce the symbol $\langle AB \rangle$ to denote the scalar term in the product, we find that

$$p_0 = \langle AB \rangle = \langle BA \rangle$$

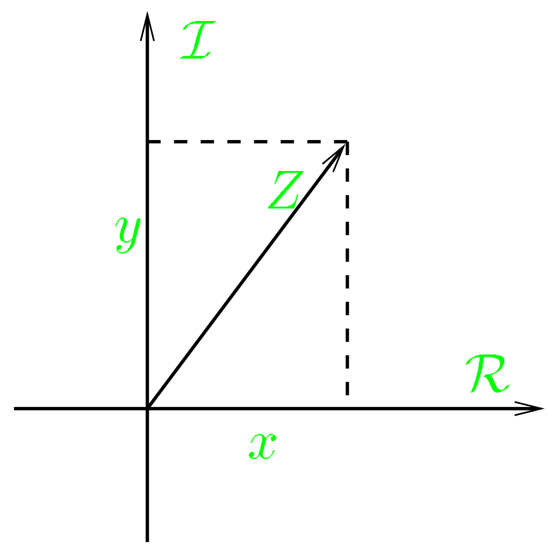
In general, however, $AB \neq BA$.

Complex Numbers and $\mathcal{G}_2$

It is clear that there is a close relationship between GA in 2-d, and the algebra of complex numbers. The unit bivector squares to -1 and generates rotations through 90°. The combination of a scalar and a bivector, which is formed naturally via the geometric product, can therefore be viewed as a complex number. We can write

$$Z = x + ye_1e_2 = x + Iy$$

Complex numbers serve a dual purpose in 2-d. They generate rotations and dilations through their polar decomposition $r\exp(j\theta)$, and they also represent vectors as points on the argand diagram.



But in $\mathcal{G}_2$ our vectors are grade-1 objects.

$$r = xe_1 + ye_2$$

Is there a natural map between this and the complex number

Z? The answer is simple – pre-multiply by e_1,

$$e_1 r = x + y e_1 e_2 = x + I y = Z$$

That is all there is to it! The role of the preferred vector e_1 is clear — it is the real axis. This product maps vectors in $\mathcal{G}_2$ onto complex numbers in a natural manner.

Complex numbers to play two roles, as rotations/dilations, and as position vectors. GA separates these roles, which is crucial to generalising complex analysis to higher dimensions.

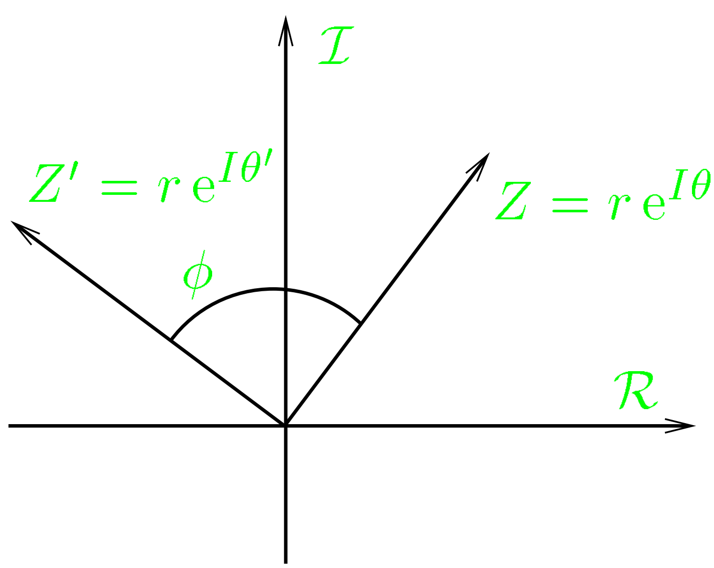
Rotations

A positive rotation through an angle ϕ for a complex number Z is achieved by

$$Z \mapsto Z'$$

$$Z' = r\,\mathrm{e}^{I(\theta + \phi)}$$

$$= \mathrm{e}^{I\phi} Z$$



We continue to use I for the unit imaginary. The exponential of a multivector is defined by power series in the normal way.

We can now apply this to rotate the vector r

$$r = e_1 Z \mapsto r' = e_1 Z'$$
$$r' = e_1 \, \mathrm{e}^{I\phi} \, Z = \, \mathrm{e}^{-I\phi} \, e_1 Z = \, \mathrm{e}^{-I\phi} \, r$$

We therefore arrive at the formulae

$$r' = \, \mathrm{e}^{-I\phi} \, r = r \, \mathrm{e}^{I\phi} \, = \, \mathrm{e}^{-I\phi/2} \, r \, \mathrm{e}^{I\phi/2}$$

which are all equivalent. The final form will turn out to be the most general. Note the importance of the fact that I anticommutes with vectors. Do not get this with complex numbers alone.

Application — Kepler Orbits

As an application of the preceeding, we will discuss an alternative formulation for 2-d motion. We start by writing the position vector x in terms of a complex number U by

$$x = U e_1 \tilde{U} = U^2 e_1, \qquad |x| = r = U\tilde{U}$$

We use the tilde for complex conjugation. Now have

$$\dot{x} = 2\dot{U} U e_1$$
$$\Rightarrow 2r\dot{U} = \dot{x} e_1 \tilde{U} = \dot{x} U e_1$$

We now introduce the new variable s defined by

$$\frac{d}{ds} = r\frac{d}{dt}, \qquad \frac{dt}{ds} = r$$

In terms of this

$$2\frac{dU}{ds} = \dot{x}Ue_1$$

and

$$2\frac{d^2U}{ds^2} = r\ddot{x}Ue_1 + \dot{x}\frac{dU}{ds}e_1$$
$$= \ddot{x}Ue_1\tilde{U}U + \tfrac{1}{2}\dot{x}^2Ue_1{}^2 = U(\ddot{x}x + \tfrac{1}{2}\dot{x}^2)$$

Now suppose we have motion in a central inverse square force:

$$m\ddot{x} = -\mu\frac{x}{r^3}$$

The equation for U becomes

$$\frac{d^2U}{ds^2} = \frac{1}{2m}U\left(\tfrac{1}{2}m\dot{x}^2 - \frac{\mu}{r}\right) = \frac{E}{2m}U$$

We recover the equation of simple harmonic motion! This has a number of advantages:

1. Easy to solve.
2. Linear, so much better for perturbation theory.
3. No singularity at $r = 0$, so better numerical stability.
4. Universal – holds for $E > 0$ and $E < 0$.
5. Extends to 3-d

The particle completes 2 cycles every time U completes one, with U is ‘centered’ on the orgin, instead of at the focus of the ellipse.