January 19, 1999

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 2

SUMMARY

In this lecture we will introduce the geometric algebra of 3D space, and start to explore some of its features. this will enable us to build up a picture of how geometric algebra can be employed to solve interesting physical problems in geometry and mechanics.

1. The geometric algebra of 3D space.
2. Planes, volumes, and the vector cross product rediscovered.
3. Rotations in 3D. Geometric Algebra provides a very clear and compact method for encoding rotations, which is considerably more powerful than working with matrices.
4. Angular momentum as a bivector.
5. A new formulation of rigid body dynamics; leading to a simplified treatment of a spinning top.

Geometric Algebra in 3-D

In Lecture 1 introduced GA in 2-d, spanned by

1	$\{e_1, e_2\}$	$e_1 \wedge e_2$
1 scalar	2 vectors	1 bivector

Now add a third vector e_3, orthogonal to e_1 and e_2. Generate 3 independent bivectors

$$e_1 e_2, \quad e_2 e_3, \quad e_3 e_1$$

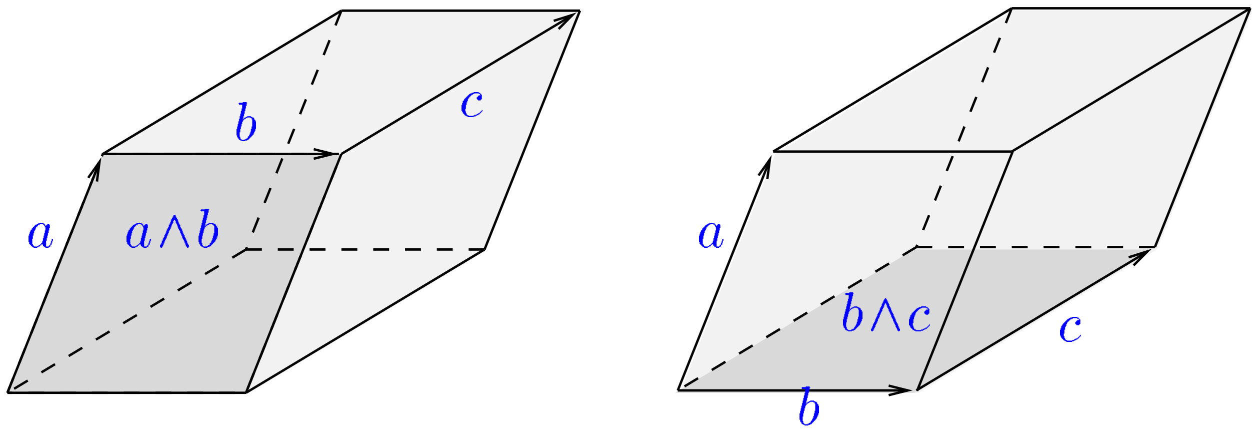
The expected number of independent planes in 3D space.

Various new products to consider. The first is a vector with an orthogonal bivector

$$(e_1 \wedge e_2) e_3 = e_1 e_2 e_3$$

The result is a trivector, the volume formed by sweeping $e_1 \wedge e_2$ along e_3. This has grade-3, as it is constructed from 3 independent vectors. Continue to use the wedge symbol for the operation of sweeping one element along another. 3 main properties

1. Associative. $(a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b \wedge c$. Same trivector formed by sweeping $a \wedge b$ along c, or $b \wedge c$ along a.



2. Antisymmetric. $a \wedge b \wedge c = -b \wedge a \wedge c = c \wedge a \wedge b$, *etc.* Swapping any two vectors reverses the orientation.

3. $a \wedge b \wedge \cdots \wedge c = 0$ if the set $a, b, \ldots c$ are linearly dependent. Follows from 2.

The outer product extends to define further higher grade quantities. In 3-d no further independent vectors, so trivectors are unique up to scale (volume), and handedness (sign). The full algebra is spanned by

1	$\{e_i\}$	$\{e_i \wedge e_j\}$	$e_1 \wedge e_2 \wedge e_3$
1 scalar	3 vectors	3 bivectors	1 trivector

Define a linear space of dimension $8 = 2^3$ (do not confuse dimension and grade!). Call this $\mathcal{G}_3$.

The size of each space is given by the binomial coefficients.

Vectors and Bivectors

The basis bivectors satisfy

$$(e_1e_2)^2 = (e_2e_3)^2 = (e_3e_1)^2 = -1$$

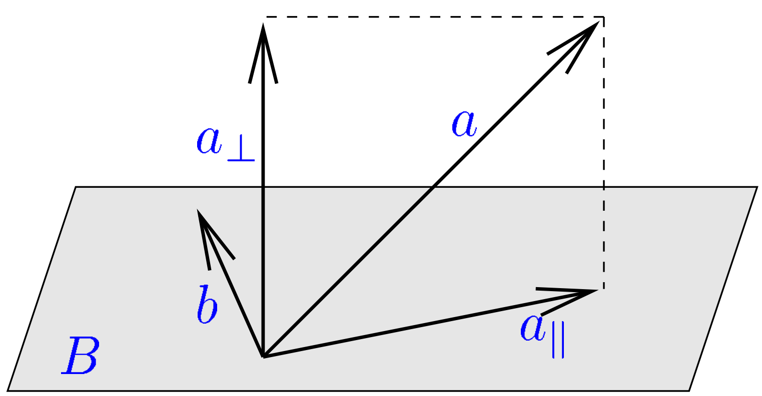
and each generates 90^{o} rotations in its own plane. (Lecture 1). Seen that

$$(e_1 \wedge e_2)e_2 = e_1e_2e_2 = e_1 \qquad \text{Vector}$$

$$(e_1 \wedge e_2)e_3 = e_1e_2e_3 = e_1 \wedge e_2 \wedge e_3 \qquad \text{Trivector}$$

The geometric product aB will contain vector and trivector parts. Decompose into

$$a = a_{||} + a_{\perp}$$



Now write $aB = (a_{||} + a_{\perp})B$. Also write

$$B = a_{||} \wedge b$$

with b orthogonal to $a_{||}$ in the B plane.

Now see that

$$a_{||}B = a_{||}(a_{||}\wedge b) = a_{||}(a_{||}b) = {a_{||}}^2 b \qquad \text{a vector}$$
$$a_{\perp}B = a_{\perp}(a_{||}\wedge b) = a_{\perp}\wedge a_{||}\wedge b \qquad \text{a trivector}$$

Write

$$aB = a\cdot B + a\wedge B$$

with dot and wedge generalised to mean lowest and highest grade part of the result. See that

$$a\cdot B = {a_{||}}^2 b = -(a_{||}b)a_{||} = -B\cdot a$$

so is antisymmetric. $a\cdot B$ projects onto the component of a in the plane, rotates this through 90^{o} and dilates by $|B|$.

Similarly

$$a\wedge B = a_{\perp}\wedge a_{||}\wedge b = a_{||}\wedge b\wedge a_{\perp} = B\wedge a$$

so is symmetric. The $a\wedge B$ term projects onto the component perpendicular to the plane, and returns a trivector.

Separate vector and trivector terms wrapped up in the invertible geometric product aB. Can can now write the dot and wedge products in terms of the geometric product

$$\begin{aligned} a\cdot B &= \tfrac{1}{2}(aB - Ba) \\ a\wedge B &= \tfrac{1}{2}(aB + Ba) \end{aligned}$$

THE BIVECTOR ALGEBRA

A further new product to consider, between independent bivectors. Find, for example,

$$(e_1 \wedge e_2)(e_2 \wedge e_3) = e_1 e_2 e_2 e_3 = e_1 e_3$$

another bivector. Also find

$$(e_2 \wedge e_3)(e_1 \wedge e_2) = e_3 e_2 e_2 e_1 = e_3 e_1 = -e_1 e_3$$

so antisymmetric, because the planes are 90° apart. Introduce the labelling scheme:

$$B_1 = e_2 e_3, \quad B_2 = e_3 e_1, \quad B_3 = e_1 e_2$$

The bivector commutator satisfies

$$B_i B_j - B_j B_i = -2\epsilon_{ijk} B_k$$

Closely linked to 3D rotations, (*cf* quantum theory of angular momentum). The commutator of 2 bivectors always results in a third bivector (or zero). They form a closed algebra. (More later ...) Also have

$$B_1{}^2 = B_2{}^2 = B_3{}^2 = -1, \quad B_1 B_2 = -B_2 B_1, \quad \textit{etc.}$$

Recovers the quaternion algebra $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$, $\boldsymbol{ij} = -\boldsymbol{ji}$. Quaternions were bivectors all along!

THE TRIVECTOR

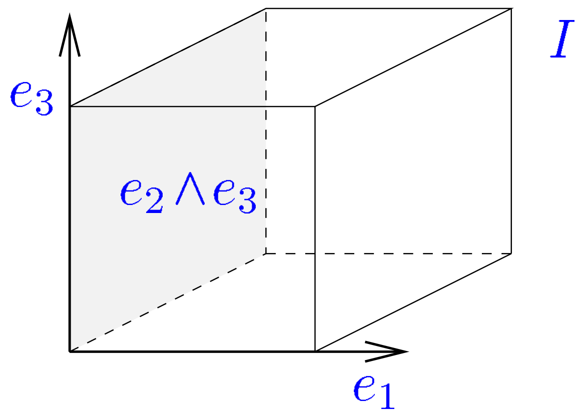
The highest grade element in 3-d algebra. Call this the pseudoscalar (or directed volume element). Write

$$I = e_1 e_2 e_3$$

Defined, by convention, to be right-handed. *i.e.* the frame $\{e_1, e_2, e_3\}$ is right-handed.

Form the product of a vector and the pseudoscalar,

$$\begin{aligned} e_1 I &= e_1(e_1 e_2 e_3) \\ &= e_2 e_3 \end{aligned}$$



returns a bivector — the plane perpendicular to the original vector. Product of a vector (grade-1) with the pseudoscalar (grade-3) is a bivector (grade-2). Call this a duality transformation.

Multiplying from the left, find that

$$I e_1 = e_1 e_2 e_3 e_1 = -e_1 e_2 e_1 e_3 = e_2 e_3$$

Result is independent of order — the pseudoscalar commutes with all vectors in 3-d, $a = aI$, so I commutes with all elements in the algebra. Always true in odd dimensions. In even dimensions I anti-commutes with vectors.

Can now express the basis bivectors in terms of their dual vectors

$$e_1 e_2 = I e_3, \qquad e_2 e_3 = I e_1, \qquad e_3 e_1 = I e_2$$

Again write

$$aI = a \cdot I$$

The dot denotes the lowest grade term in the product. This is a projection — projecting onto the component of I perpendicular to a.

Next form the square of the pseudoscalar

$$I^2 = e_1 e_2 e_3 e_1 e_2 e_3 = e_1 e_2 e_1 e_2 = -1$$

The pseudoscalar commutes with all elements and squares to -1. Another candidate for a unit imaginary. Correct choice depends on context — it is dictated by the physics. Makes GA a much richer language.

Finally, form the product of a bivector and the pseudoscalar:

$$I(e_1 \wedge e_2) = I e_1 e_2 e_3 e_3 = I I e_3 = -e_3$$

Get minus the vector perpendicular to the $e_1 \wedge e_2$ plane. Can recover the 3-d vector cross product

$$a \boldsymbol{\times} b = -I(a \wedge b) = -I\, a \wedge b$$

(a bold cross — need the $\times$ for something more useful.) A bivector in disguise! Replace 'axial vectors' with the more natural idea of a bivector. Can only replace bivectors with dual vectors like this in 3-d.

NB have introduced an operator ordering convention: in the absence of brackets, *dot and wedge products are performed before geometric products.*

REVERSION

Important operation in GA — reverse the order of vectors in any product. Denoted with a tilde, $\tilde{A}$. Scalars and vectors are unchanged, bivectors and trivectors change sign

$$(e_1 e_2)^{\sim} = e_2 e_1 = -e_1 e_2$$

$$\tilde{I} = e_3 e_2 e_1 = e_1 e_3 e_2 = -e_1 e_2 e_3 = -I$$

Summarise by writing a general 3-d multivector

$$M = \alpha + a + B + \beta I.$$

From the above, get

$$\tilde{M} = \alpha + a - B - \beta I$$

Aside — Quantum Spin

The full geometric product for vectors gives

$$e_i e_j = e_i \cdot e_j + e_i \wedge e_j = \delta_{ij} + I\epsilon_{ijk} e_k$$

should be familiar - it is the Pauli algebra of quantum mechanics! Suggests that the matrix structure of quantum spin has a more geometric origin. Has generated some controversy over the role of matrix operators.

The Pauli matrices form a matrix representation of $\mathcal{G}_3$. Can view the algebra this way, but matrix manipulations are slow.

Rotations

In Lecture 1, found that a vector in 2-d is rotated through θ in the $e_1 e_2$ plane by one of

$$\begin{aligned} a \mapsto a' &= \mathrm{e}^{-e_1 e_2 \theta} a = a\, \mathrm{e}^{e_1 e_2 \theta} \\ &= \mathrm{e}^{-e_1 e_2 \theta/2} a\, \mathrm{e}^{e_1 e_2 \theta/2} \end{aligned}$$

Want to find a 3-d version of this. Any of the above works for a vector in the $e_1 e_2$ plane, but also require that e_3 is unchanged. (This is the axis — an entirely 3-d idea.)

Key is to recall that e_3 commutes with $e_1 e_2$, so

$$\begin{aligned} \mathrm{e}^{-e_1 e_2 \theta} e_3 &= [\cos(\theta) - \sin(\theta) e_1 e_2] e_3 \\ &= e_3 (\cos(\theta) - \sin(\theta) e_1 e_2) = e_3 \, \mathrm{e}^{-e_1 e_2 \theta} \end{aligned}$$

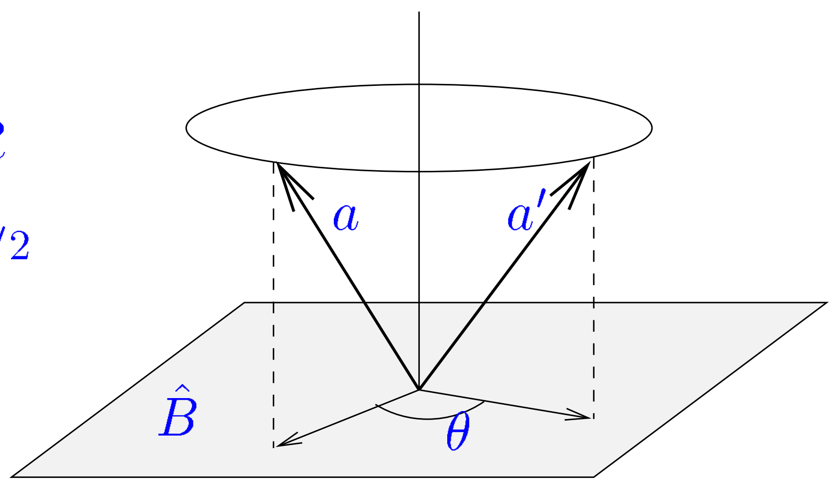
Clear that only the intermediate, double-sided formula leaves vectors perpendicular to the plane untouched:

$$\mathrm{e}^{-e_1 e_2 \theta/2} e_3 \, \mathrm{e}^{e_1 e_2 \theta/2} = e_3 \, \mathrm{e}^{-e_1 e_2 \theta/2} \, \mathrm{e}^{e_1 e_2 \theta/2} = e_3$$

A vector is rotated through θ in the $\hat{B}$ plane ($\hat{B}^2 = -1$) by

$$a' = R a \tilde{R}$$

$$R = \mathrm{e}^{-\hat{B}\theta/2}$$



R is called a rotor. It satisfies the normalisation condition

$$R\tilde{R} = \tilde{R}R = 1$$

R is formed from the scalar + bivector algebra (the quaternions again!) with one constraint, leaving 3 degrees of freedom, as expected.

What about bivectors, how do we rotate these?

$$\begin{aligned} B' &= a' \wedge b' = \tfrac{1}{2}(a'b' - b'a') \\ &= \tfrac{1}{2}(Ra\tilde{R}Rb\tilde{R} - Rb\tilde{R}Ra\tilde{R}) \\ &= \tfrac{1}{2}R(ab - ba)\tilde{R} = Ra\wedge b\tilde{R} = RB\tilde{R}. \end{aligned}$$

The same formula as vectors! True for all multivectors. One of the most attractive features of geometric algebra.

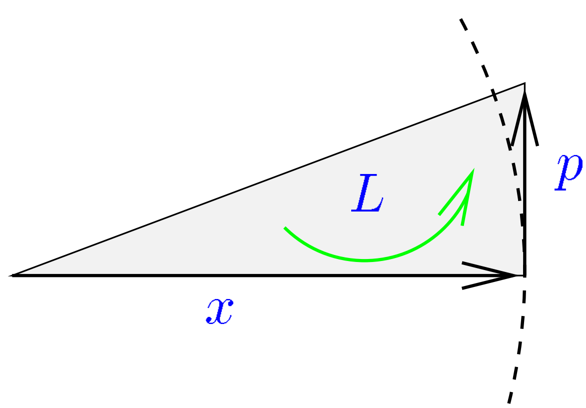
Angular Momentum

Usually define angular momentum by

$$L = x \times p.$$

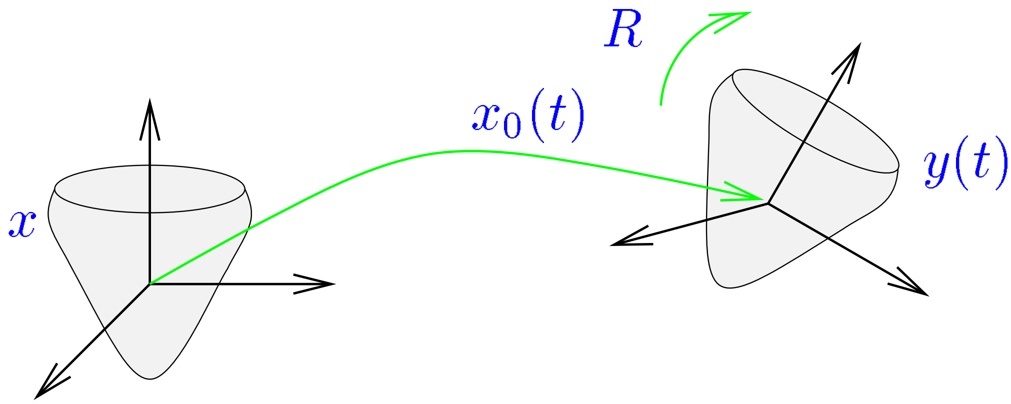
But we have a better alternative. We understand angular momentum in terms of a particle sweeping out a plane. Therefore define angular momentum as the bivector $L = x \wedge p$. This now works in 2-d and 4-d as well.

The particle sweeps out the plane $L = x \wedge p$



Rigid-Body Dynamics

Our first major application of GA. Have a rigid body moving through space. Relate the vector position of points in the moving body $y(t)$ back to a fixed 'reference' body.



x_0 is the position in space of the centre of mass. Have

$$y(t) = R(t)x\tilde{R}(t) + x_0(t)$$

Places the rotational motion in the time-dependent rotor $R(t)$.

Next need an expression for the angular velocity. This must be a bivector as well. Suppose the frame $\{f_k\}$ is rotating in space. Relate to a fixed orthonormal frame $\{e_k\}$ by

$$f_k = R(t)e_k\tilde{R}(t)$$

Angular momentum vector ω is usually defined by

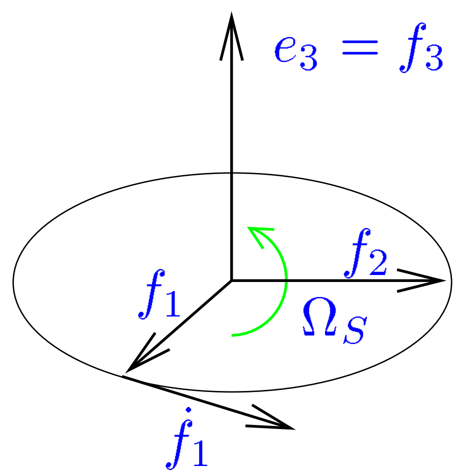
$$\dot{f}_k = \omega \times f_k = -I\,\omega \wedge f_k = (-I\omega)\cdot f_k$$

Introduce the (space) angular-velocity bivector

$$\Omega_S = I\omega$$

This has the correct orientation.

Ω_S has the orientation of $f_1 \wedge \dot{f}_1$, so is $+e_1 \wedge e_2$ when $\omega = e_3$.



Next look at the time dependence.

$$\dot{f}_k = \dot{R} e_k \tilde{R} + R e_k \dot{\tilde{R}} = \dot{R}\tilde{R} f_k + f_k R \dot{\tilde{R}}$$

But $R\tilde{R} = 1$, so

$$0 = \partial_t (R\tilde{R}) = \dot{R}\tilde{R} + R\dot{\tilde{R}}$$

So

$$\dot{R}\tilde{R} = -R\dot{\tilde{R}} = -(\dot{R}\tilde{R})^{\sim}$$

So $\dot{R}\tilde{R}$ is even and equal to minus its own reverse. It must be a pure bivector. Now get

$$\dot{f}_k = \dot{R}\tilde{R} f_k - f_k \dot{R}\tilde{R} = (2\dot{R}\tilde{R}) \cdot f_k$$

See that $2\dot{R}\tilde{R} = -\Omega_S$.

Dynamics reduces to the single rotor equation

$$\dot{R} = -\tfrac{1}{2}\Omega_S R \quad \text{or} \quad \dot{\tilde{R}} = \tfrac{1}{2}\tilde{R}\Omega_S$$

Equations like this are very common in physics. Will encounter many examples.

Can also express in terms of the body angular velocities, Ω_B, *i.e.* Ω_S expressed back in the 'reference' copy

$$\Omega_S = R\Omega_B\tilde{R}, \quad \Omega_B = \tilde{R}\Omega_S R$$

In terms of these we have

$$\dot{R} = -\tfrac{1}{2}R\Omega_B, \quad \text{and} \quad \dot{\tilde{R}} = \tfrac{1}{2}\Omega_B\tilde{R}$$