January 21, 1999

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 3

SUMMARY

This lecture is split into three sections. In the first we will conclude our treatment of rigid body dynamics by solving the equations of motion for a symmetric top. In the second section we will put some of the ideas from the first two lectures onto a firmer axiomatic basis. In the final section we will start to look at the GA treatment of reflections and rotations in greater depth.

- The inertia Tensor.
- The rotor solution for the motion of a symmetric top.
- The axioms of geometric algebra.
- An array of useful algebraic results.
- Reflections, rotations and rotors

The webpage for this course is

www.mrao.cam.ac.uk/∼clifford/ptIIIcourse/.

The Inertia Tensor

Rigid body has density ρ, so

$$\int d^3x\,\rho = m, \qquad \int d^3x\,\rho x = 0$$

The velocity of the point y is

$$\begin{aligned} v(t) &= \dot{R}x\tilde{R} + Rx\dot{\tilde{R}} + \dot{x}_0 \\ &= -\tfrac{1}{2}R\Omega_B x\tilde{R} + \tfrac{1}{2}Rx\Omega_B\tilde{R} + v_0 \\ &= R\,x\cdot\Omega_B\,\tilde{R} + v_0 \end{aligned}$$

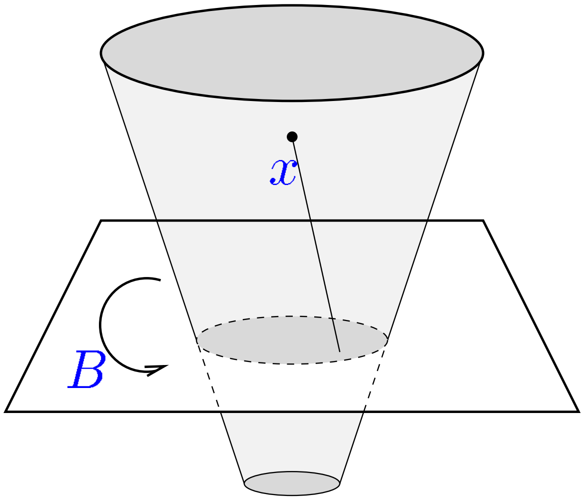
(v_0 is the velocity of the centre of mass.) We need the angular momentum bivector L

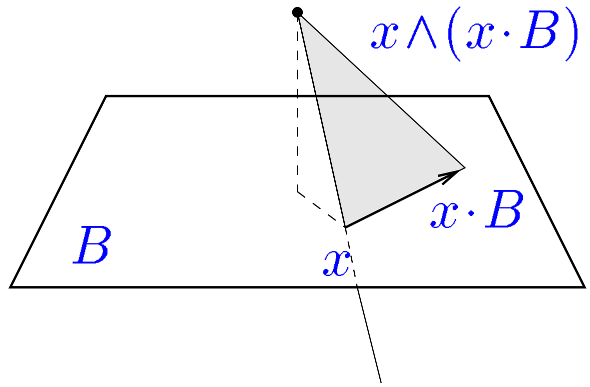
$$\begin{aligned} L &= \textstyle\int d^3x\,\rho(y - x_0)\wedge v \\ &= \textstyle\int d^3x\,\rho(Rx\tilde{R})\wedge(R\,x\cdot\Omega_B\,\tilde{R} + v_0) \\ &= R\left(\textstyle\int d^3x\,\rho x\wedge(x\cdot\Omega_B)\right)\tilde{R} \end{aligned}$$

From this we extract inertia tensor $\mathcal{I}$

$$\mathcal{I}(B) = \int d^3x\,\rho x\wedge(x\cdot B)$$

A linear function mapping bivectors to bivectors.





The body rotates in the B plane, at angular frequency $|B|$. The momentum density is $\rho x\cdot B$. Angular momentum density is $x\wedge(\rho x\cdot B)$. Integrate to get the total, $\mathcal{I}(B)$, expressed in the reference body. Rotate to

$$L = R\mathcal{I}(\Omega_B)\tilde{R}$$

$\mathcal{I}(B)$ will lie in the same plane as B if B is perpendicular to one of the principal axes

Now $\dot{L} = T$ (the couple as a bivector), so form

$$\begin{aligned}\dot{L} &= \dot{R}\mathcal{I}(\Omega_B)\tilde{R} + R\mathcal{I}(\Omega_B)\dot{\tilde{R}} + R\mathcal{I}(\dot{\Omega}_B)\tilde{R} \\ &= R[\mathcal{I}(\dot{\Omega}_B) - \tfrac{1}{2}\Omega_B\mathcal{I}(\Omega_B) + \tfrac{1}{2}\mathcal{I}(\Omega_B)\Omega_B]\tilde{R} \\ &= R[\mathcal{I}(\dot{\Omega}_B) - \Omega_B\times\mathcal{I}(\Omega_B)]\tilde{R}.\end{aligned}$$

Have introduced the extremely useful commutator product

$$A\times B = \tfrac{1}{2}(AB - BA)$$

Do not confuse with the cross product! The torque-free equation $\dot{L} = 0$ reduces to

$$\mathcal{I}(\dot{\Omega}_B) - \Omega_B \times \mathcal{I}(\Omega_B) = 0$$

Align the body frame $\{e_k\}$ with the principal axes, with moments of inertia $i_k, k = 1 \ldots 3$. Have

$$\Omega_B = \sum_k \omega_k I e_k, \qquad \Omega_S = \sum_k \omega_k I f_k$$

and

$$L = \sum_k i_k \omega_k I f_k.$$

Expanding out recovers the Euler equations, *e.g.*

$$\dot{\omega}_3 I i_3 e_3 = (\omega_1 I e_1 + \omega_2 I e_2) \times (\omega_1 i_1 I e_1 + \omega_2 i_2 I e_2)$$
$$\Rightarrow \quad i_3 \dot{\omega}_3 = (i_1 - i_2)\omega_1 \omega_2$$

Example — The Symmetric Top

Have two equal moments of inertia, $i_1 = i_2 \neq i_3$. Immediately get that ω_3 is constant. (Handout gives an alternative coordinate-free derivation). Write

$$\mathcal{I}(B) = i_1 B + (i_3 - i_1)(B \wedge e_3) e_3$$

NB $B \wedge e_3$ is a trivector. Now have

$$\begin{aligned} i_1 \Omega_B &= \mathcal{I}(\Omega_B) - (i_3 - i_1)(\Omega_B \wedge e_3) e_3 \\ &= \mathcal{I}(\Omega_B) + (i_1 - i_3)\omega_3 I e_3 \end{aligned}$$

so

$$\Omega_S = R\Omega_B \tilde{R} = \frac{1}{i_1} L + \frac{i_1 - i_3}{i_1} \omega_3 R I e_3 \tilde{R}$$

The rotor equation now becomes

$$\dot{R} = -\tfrac{1}{2}\Omega_S R = -\frac{1}{2i_1}(LR + R(i_1 - i_3)\omega_3 I e_3)$$

Define two constant precession rates,

$$\Omega_l = \frac{1}{i_1} L, \qquad \Omega_r = \omega_3 \frac{i_1 - i_3}{i_1} I e_3$$

The rotor equation is now

$$\dot{R} = -\tfrac{1}{2}\Omega_l R - \tfrac{1}{2} R \Omega_r$$

which integrates immediately to

$$R(t) = \exp(-\tfrac{1}{2}\Omega_l t) R(0) \exp(-\tfrac{1}{2}\Omega_r t)$$

Fully describes the motion of a symmetric top. An ‘internal’ rotation in the $e_1 e_2$ plane (a symmetry of the body), followed by a rotation in the angular-momentum plane.

AXIOMATIC DEVELOPMENT

We should now have an intuitive feel for the elements of a geometric algebra and some of their properties. Now need a proper axiomatic framework. Use symbol $\mathcal{G}_n$ for the GA of n-dimensional (Euclidean) space. This space is linear over the reals

$$\lambda A + \mu B \in \mathcal{G}_n, \qquad \forall \lambda, \mu \in \mathcal{R},\ A, B \in \mathcal{G}_n$$

Not interested in complex superpositions!

The linear space $\mathcal{G}_n$ is graded. Elements of this space are called multivectors. Every multivector can be written as a sum of pure grade terms

$$A = \langle A \rangle_0 + \langle A \rangle_1 + \cdots = \sum_r \langle A \rangle_r$$

The operator $\langle A \rangle_r$ projects onto the grade-r terms in A. Each graded subspace of $\mathcal{G}_n$ is also closed under addition and forms a linear subspace.

Multivectors containing terms of only one grade are called homogeneous. Write these as A_r,

$$\langle A_r \rangle_r = A_r$$

NB Avoid confusing A_r with $\{e_k\}$.

The grade-0 terms in $\mathcal{G}_n$ are real scalars. Abbreviate

$$\langle A \rangle_0 = \langle A \rangle$$

The grade-1 objects $\langle A \rangle_1$ are vectors.

The Geometric Product

Recall from Lecture 1 that the geometric product is associative

$$A(BC) = (AB)C = ABC$$

and distributive over addition

$$A(B+C) = AB + AC$$

Also the square of any vector is a scalar. From these get

$$ab + ba = (a+b)^2 - a^2 - b^2,$$

Another scalar. Define the inner product

$$a \cdot b = \tfrac{1}{2}(ab + ba)$$

and the outer product

$$a \wedge b = \tfrac{1}{2}(ab - ba)$$

Both defined from the geometric product. Recover familiar result

$$ab = a \cdot b + a \wedge b$$

Now extend this idea. Form the product of a vector and a bivector

$$\begin{aligned} a(b\wedge c) &= \tfrac{1}{2}a(bc - cb) \\ &= (a\cdot b)c - (a\cdot c)b - \tfrac{1}{2}(bac - cab) \\ &= 2(a\cdot b)c - 2(a\cdot c)b + \tfrac{1}{2}(bc - cb)a \end{aligned}$$

Define the inner product

$$a\cdot(b\wedge c) = \tfrac{1}{2}[a(b\wedge c) - (b\wedge c)a] = (a\cdot b)c - (a\cdot c)b$$

Must be a vector. The remaining symmetric part

$$a\wedge(b\wedge c) = \tfrac{1}{2}[a(b\wedge c) + (b\wedge c)a] = a\wedge b\wedge c$$

is a trivector – totally antisymmetric on a, b, c. Now have

$$a(b\wedge c) = a\cdot(b\wedge c) + a\wedge(b\wedge c)$$

Found this in Lecture 2 from a different, geometric argument.

N.B. Recall the important operator ordering convention: in the absence of brackets, *inner and outer products take precedence over geometric products*. *i.e.*

$$(a\cdot b)c = a\cdot b\, c$$

no confusion possible with $a\cdot(bc)$

Blades and Bases

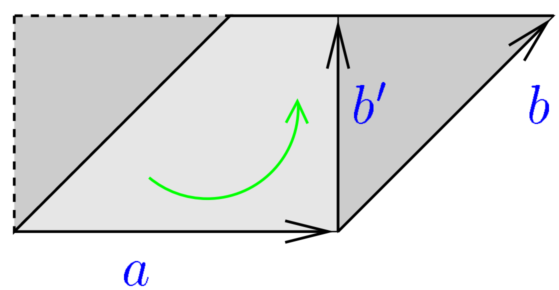
Outer product is the totally antisymmetrised sum of all products of vectors,

$$a_1 \wedge a_2 \wedge \cdots \wedge a_r = \frac{1}{r!} \sum (-1)^{\epsilon} a_{k_1} a_{k_2} \cdots a_{k_r}$$

Sum runs over every permutation of indices $k_1 \ldots k_r$. $\epsilon = \pm 1$ for even/ odd permutation. A multivector which is purely an outer product is called a blade.

Fortunately every blade can be written as a geometric product of orthogonal, anticommuting vectors. Anticommutation then imposes the antisymmetry. Take vectors a, b, $b' = b - \lambda a$

$$\begin{aligned} a \wedge b &= a \wedge (b - \lambda a) \\ &= a \wedge b' \end{aligned}$$



Same area and orientation so same bivector. Form

$$a \cdot b' = a \cdot (b - \lambda a) = a \cdot b - \lambda a^2.$$

Set $\lambda = a \cdot b / a^2$ so that $a \cdot b' = 0$. Can write

$$a \wedge b = a \wedge b' = ab'.$$

Full proof continues by induction. Note that

$$b' = b - a^{-1}a\cdot b = b - \tfrac{1}{2}a^{-1}(ab+ba)$$
$$= \tfrac{1}{2}(b - a^{-1}ba) = a^{-1}\tfrac{1}{2}(ab-ba) = a^{-1}a\wedge b.$$

Clear why $ab' = a\wedge b$, and generalises.

Can now view $\mathcal{G}_n$ in terms of orthonormal basis vectors $\{e_i\}, i = 1\ldots n$. Build up a basis for the algebra as

$$1, \quad e_i, \quad e_ie_j\,(i<j) \quad e_ie_je_k\,(i<j<k) \qquad \textit{etc.}$$

Denote each grade-r subspace of $\mathcal{G}_n$ by $\mathcal{G}_n^r$. Natural question: what is the dimension of each of these graded subspaces?

Choose r distinct vectors. Different because of the total antisymmetry. Order is irrelevant, again because of the antisymmetry, Just need number of distinct combinations of r objects from a set of n. *i.e.*

$$\text{Dim}\,[\mathcal{G}_n^r] = \binom{r}{n}.$$

Get the binomial coefficients. Contain a surprising wealth of geometric information! The total dimension is

$$\text{Dim}\,[\mathcal{G}_n] = \sum_{r=0}^{n}\binom{r}{n} = (1+1)^n = 2^n.$$

Important Point *not all homogeneous multivectors are pure blades.* Confusing at first, need to go to 4-d for first counter-example. Take $\{e_1 \dots e_4\}$ orthonormal basis for $\mathcal{G}_4$. Six independent basis bivectors. Can construct terms like

$$B = \alpha e_1 \wedge e_2 + \beta e_3 \wedge e_4, \quad \alpha, \beta \in \mathcal{R}.$$

B is a pure bivector — homogeneous. But cannot find two vectors a and b such that $B = a \wedge b$. Because $e_1 \wedge e_2$ and $e_3 \wedge e_4$ do not share a common line. Makes the bivector B hard to visualise. An alternative is provided by projective geometry (non-intersecting lines).

Further Properties

Take a grade-r blade, decomposed into orthogonal vectors $a_1 a_2 \cdots a_r$. Have

$$\begin{aligned} a a_1 a_2 \cdots a_r &= 2a \cdot a_1\, a_2 \cdots a_r - a_1 a a_2 \cdots a_r \\ &= 2 \sum_{k=1}^{r} (-1)^{k+1} a \cdot a_k\, a_1 a_2 \cdots \check{a}_k \cdots a_r \\ &\quad + (-1)^r a_1 a_2 \cdots a_r a \end{aligned}$$

The $\check{a}_k$ term is missing from the series. Each term in the sum has grade $r-1$, so define

$$a \cdot A_r = \langle a A_r \rangle_{r-1} = \tfrac{1}{2}(a A_r - (-1)^r A_r a)$$

Remaining term in aA_r is totally antisymmetric, so have

$$a\wedge A_r = \langle aA_r\rangle_{r+1} = \tfrac{1}{2}(aA_r + (-1)^r A_r a)$$

Can still write

$$aA_r = a\cdot A_r + a\wedge A_r.$$

Multiplication by a vector raises and lowers the grade by 1.

Now suppose the $\{a_i\}$ are arbitrary. Write

$$\begin{aligned} &a\cdot(a_1\wedge a_2\wedge\cdots\wedge a_r) \\ &\quad= \tfrac{1}{2}[a\langle a_1a_2\cdots a_r\rangle_r - (-1)^r\langle a_1a_2\cdots a_r\rangle_r a] \\ &\quad= \tfrac{1}{2}\langle aa_1a_2\cdots a_r - (-1)^r a_1a_2\cdots a_r a\rangle_{r-1} \end{aligned}$$

Final step because

$$a_1a_2\cdots a_r = A_r + A_{r-2} + \cdots$$

Only the A_{r-2} term is a potential problem, but

$$\tfrac{1}{2}(aA_{r-2} - (-1)^r A_{r-2}a) = a\cdot A_{r-2}$$

is grade $r-3$. Now use preceding to get

$$\begin{aligned} &a\cdot(a_1\wedge a_2\wedge\cdots\wedge a_r) \\ &\quad= \langle\textstyle\sum_{k=1}^r (-1)^{k+1} a\cdot a_k\, a_1a_2\cdots\check{a}_k\cdots a_r\rangle_{r-1} \\ &\quad= \textstyle\sum_{k=1}^r (-1)^{k+1} a\cdot a_k\, a_1\wedge a_2\wedge\cdots\wedge\check{a}_k\wedge\cdots\wedge a_r \end{aligned}$$

Extremely useful! First two cases

$$
\begin{aligned}
a\cdot(a_1\wedge a_2) &= a\cdot a_1\, a_2 - a\cdot a_2\, a_1 \\
a\cdot(a_1\wedge a_2\wedge a_3) &= a\cdot a_1\, a_2\wedge a_3 - a\cdot a_2\, a_1\wedge a_3 \\
&\quad + a\cdot a_3\, a_1\wedge a_2
\end{aligned}
$$

NB similarity with double cross product of vectors in 3-d.

The general product of two homogeneous multivectors decomposes as

$$
\begin{aligned}
A_r B_s &= \langle A_r B_s\rangle_{|r-s|} + \langle A_r B_s\rangle_{|r-s|+2} + \cdots \\
&\quad + \langle A_r B_s\rangle_{r+s}
\end{aligned}
$$

Can see this by expanding both out in terms of an orthogonal basis. Retain the $\cdot$ and $\wedge$ symbols for the lowest and highest grade terms in this series

$$
\begin{aligned}
A_r\cdot B_s &= \langle A_r B_s\rangle_{|r-s|} \\
A_r\wedge B_s &= \langle A_r B_s\rangle_{r+s}
\end{aligned}
$$

Definitions ensure the exterior product is also associative.