January 26, 1999

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 4

SUMMARY

In this lecture we will build on the idea of rotations represented by rotors, and use this to explore topics in the theory of Lie groups and Lie algebras.

1. Reflections, Rotations and the rotor description.
2. Rotor groups, multivector transformations and 'spin-1/2'
3. Lie Groups, continuous groups and the manifold structure of rotors in 3-d.
4. Bivector generators and Lie algebras.
5. Complex structures and doubling bivectors.
6. Unitary groups expressed in real geometric algebra.

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Pseudoscalars and Duality

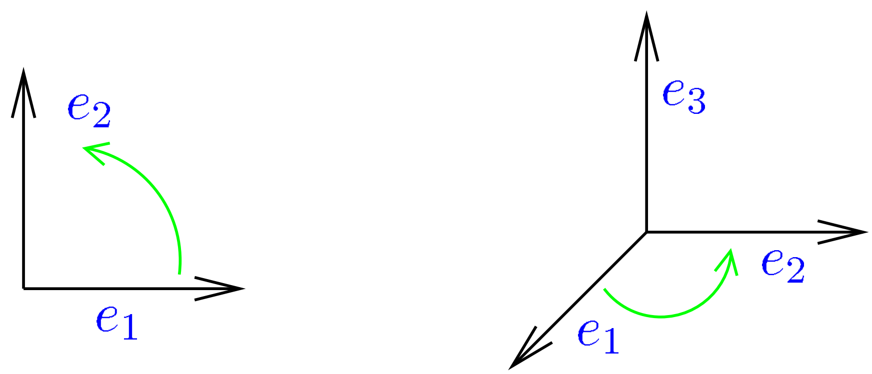
Exterior product of n vectors in $\mathcal{G}_n$ gives a multiple of the pseudoscalar, I. Two key properties:

1. Normalisation

$$|I^2| = 1.$$

Sign of I^2 depends on dimension (and signature).

2. Right-Handed.



$e_1 \wedge e_2$ is right handed, by definition. $e_1 \wedge e_2 \wedge e_3$ is if looking down e_3 gives right-handed plane. Continue inductively.

Product of the grade-n pseudoscalar I with grade-r multivector A_r is a grade $n-r$ multivector

$$IA_r = \langle IA_r \rangle_{n-r}$$

Called a duality transformation. If A_r is a blade, get the orthogonal complement of A_r – the blade formed from the space of vectors not contained in A_r.

Summarise the commutative properties of I by

$$IA_r = (-1)^{r(n-1)} A_r I$$

I always commutes with even grade. For odd grade depends on dimension of space.

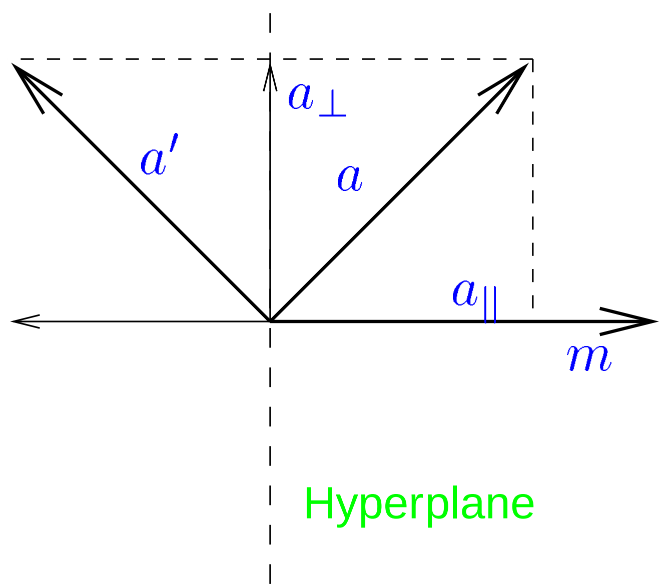
Important use for I: interchanging dot and wedge products. Take A_r and B_s, $r+s \leq n$,

$$A_r \cdot (B_s I) = \langle A_r B_s I \rangle_{|r-(n-s)|} = \langle A_r B_s I \rangle_{n-(r+s)}$$
$$= \langle A_r B_s \rangle_{r+s} I = A_r \wedge B_s \, I$$

Have already used this in 3-d.

Reflections

$$a = a_\perp + a_{||}$$
$$a' = a_\perp - a_{||}$$



Reflect the vector a in the (hyper)plane orthogonal to unit vector m.

Component of a parallel to m changes sign, perpendicular component is unchanged. Parallel component is the projection onto m:

$$a_{||} = a\cdot m\, m$$

The perpendicular component is the remainder

$$a_{\perp} = a - a\cdot m\, m = (am - a\cdot m)m = a\wedge m\, m$$

Shows how the wedge product projects out component perpendicular to a vector.

The reflection gives

$$\begin{aligned} a' &= -a\cdot m\, m + a\wedge m\, m \\ &= -(m\cdot a + m\wedge a)m = -mam \end{aligned}$$

A remarkably neat formula!

Simple to check the desired properties. For a vector parallel to m

$$-m(\lambda m)m = -\lambda mmm = -\lambda m$$

transformed to minus itself. For vector perpendicular to m

$$-m(n)m = -mnm = nmm = n \qquad (n\cdot m = 0)$$

so unchanged. Also give a simple proof that lengths and angles unchanged

$$a'b' = (-mam)(-mbm) = mabm$$

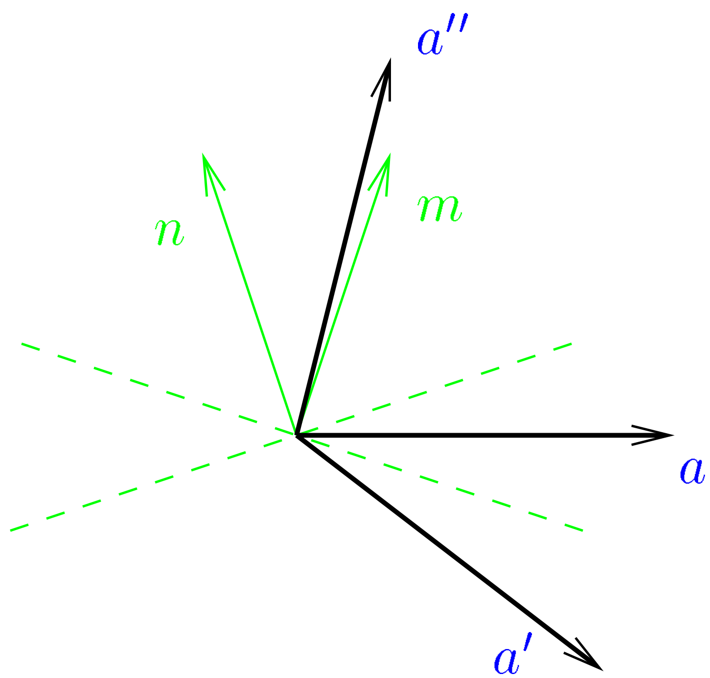
Scalar part gives $a' \cdot b' = m\, a \cdot b\, m = a \cdot b$, as expected.

Bivector part gives

$$a' \wedge b' = m\, a \wedge b\, m$$

A crucial sign change cf vectors. Origin of distinction between polar and axial vectors.

Rotations



Theorem: *Successive reflections generated by two vectors* m *and* n *gives a rotation in the* $m \wedge n$ *plane*.

a' is the result of reflecting a in the plane perpendicular to m

a'' is the result of reflecting a' in the plane perpendicular to n.

Component of a outside the plane is untouched.

Simple trigonometry: angle between a and a'' is twice angle between m and n, so rotate through 2θ in the $m\wedge n$ plane, where $m\cdot n = \cos(\theta)$.

How does this look in GA?

$$a' = -mam$$

$$a'' = -na'n = -n(-mam)n = nmamn$$

This is beginning to look very simple! We define

$$R = nm, \qquad \tilde{R} = mn$$

Note the geometric product. We can now write a rotation as

$$a \mapsto Ra\tilde{R}$$

Incredibly, works for any grade of multivector, in any dimension, of any signature! As seen already, R is a rotor. Now make contact with bivector approach.

R is the geometric product of two unit vectors n and m,

$$R = nm = n\cdot m + n\wedge m = \cos(\theta) + n\wedge m$$

What is the magnitude of the bivector $n\wedge m$?

$$(n\wedge m)\cdot(n\wedge m) = \langle n\wedge m\, n\wedge m\rangle = \langle nm\, n\wedge m\rangle$$

$$= \langle nm(n\cdot m - mn)\rangle = \cos^2(\theta) - 1 = -\sin^2(\theta)$$

Define a unit bivector in the $m\wedge n$ plane by

$$\hat{B} = m\wedge n/\sin(\theta), \qquad \hat{B}^2 = -1$$

NB Using the correct right-handed orientation for $\hat{B}$, as θ defined as angle between m and n in positive sense from m to n. Now have

$$R = \cos(\theta) - \hat{B}\sin(\theta)$$

Familiar? it is the polar decomposition of a complex number back again. Unit imaginary replaced by the unit bivector $\hat{B}$. Write

$$R = \exp\{-\hat{B}\theta\}$$

(Exponential defined by power series – this always converges for multivectors).

But our formula was for a rotation through 2θ. For rotation through θ, need the half angle formula

$$R = \exp\{-\hat{B}\theta/2\}$$

which gives

$$a \mapsto \mathrm{e}^{-\hat{B}\theta/2} a \, \mathrm{e}^{\hat{B}\theta/2}$$

for a positive rotation through θ in the $\hat{B}$ plane. In GA think of rotations taking place in a plane not around an axis.

The Rotor Group

Form composite of two rotations

$$a \mapsto R_2(R_1 a \tilde{R}_1)\tilde{R}_2 = R_2 R_1 a \tilde{R}_1 \tilde{R}_2$$

Define

$$R = R_2 R_1$$

Still have $a \mapsto Ra\tilde{R}$. R is a geometric products of an even number of unit vectors,

$$R = kl \cdots mn$$

This defines a rotor. The reversed rotor is

$$\tilde{R} = nm \cdots lk$$

so still have normalisation condition

$$R\tilde{R} = kl \cdots mnnm \cdots lk = 1 = \tilde{R}R$$

In non-Euclidean spaces require this.

Now decompose general multivector into sum of blades. Write each as product of orthogonal vectors. Take

$$A_r = a_1 a_2 \cdots a_r$$

Rotate each vector to $a'_i = Ra_i\tilde{R}$. Get

$$A'_r = a'_1 a'_2 \cdots a'_r = Ra_1\tilde{R}\, Ra_2\tilde{R} \cdots Ra_r\tilde{R}$$
$$= Ra_1 a_2 \cdots a_r\tilde{R} = RA_r\tilde{R}$$

Recover the same law as for vectors! All multivectors transform same way when component vectors are rotated.

Spin-1/2

Have rotors R and $R_\theta = \exp(-\hat{B}\theta/2)$. Product rotor is

$$R' = R_\theta R = \mathrm{e}^{-\hat{B}\theta/2}\, R.$$

Now increase θ from 0 through to 2π. $\theta = 2\pi$ is identity operation. But R transforms to

$$R \mapsto R' = \mathrm{e}^{-\hat{B}\pi} R = (\cos\pi - \hat{B}\sin\pi)R = -R.$$

Rotors change sign under 360° rotations. Just like fermions in quantum theory! But no quantum mechanics here. Can see effect with coupled rotations.

LIE GROUPS

Rotors form an infinite-dimensional continuous group — a Lie group. Not a vector space, actually a manifold. See this in 3-d. Write

$$R = x_0 + x_1 Ie_1 + x_2 Ie_2 + x_3 Ie_3$$

Then

$$R\tilde{R} = {x_0}^2 + {x_1}^2 + {x_2}^2 + {x_3}^2 = 1$$

Defines a unit vector in 4-d. Group manifold is a 3-sphere. Not same as rotation group.

Rotations are formed by $a \mapsto Ra\tilde{R}$, so R and $-R$ give same rotation. Rotation group manifold is 3-sphere with opposite points identified.

Attitude of a rigid body described by a rotor, so configuration space for rigid body dynamics is a 3-sphere. Important for

1. Finding best fit rotation.
2. Extrapolating between two rotations.
3. Lagrangian treatment and conjugate momenta.
4. Quantum rigid rotor.

Abstract idea. Lie group = Manifold $+$ product $z = \phi(x, y)$.

BIVECTOR GENERATORS

Question: can any rotor be written as the exponential of a bivector? Define a one-parameter 'Abelian subgroup' $R(\lambda)$

$$R(\lambda + \mu) = R(\lambda)R(\mu)$$

Must have $R(0) = 1$. Look at vector

$$a(\lambda) = Ra_0\tilde{R}$$

Differentiate with respect to λ to get

$$a'(\lambda) = R' a_0 \tilde{R} + R a_0 \tilde{R}' = R'\tilde{R} a - a R'\tilde{R}$$

Have used familiar result

$$(R\tilde{R})' = 0 = R'\tilde{R} + R\tilde{R}'$$

$R'\tilde{R}$ could have grades 2, 6, 10 *etc*. But commutator product with any vector is a vector, so $R'\tilde{R}$ is a bivector only,

$$R'(\lambda) = -\tfrac{1}{2} B(\lambda) R(\lambda)$$

But have

$$\begin{aligned} R'(\lambda+\mu) &= -\tfrac{1}{2} B(\lambda+\mu) R(\lambda+\mu) \\ &= -\tfrac{1}{2} B(\lambda+\mu) R(\lambda) R(\mu) \\ &= [R(\lambda)R(\mu)]' = -\tfrac{1}{2} B(\lambda) R(\lambda) R(\mu) \end{aligned}$$

So B is constant along this curve. Integrate to get

$$R(\lambda) = \mathrm{e}^{-\lambda B/2}$$

Any rotor on this curve is exponential of a bivector. Manifold idea $\Rightarrow$ true for all near identity.

Euclidean space: All rotors $=$ bivector exponential

Mixed Signature: $R(\lambda) = \pm\,\mathrm{e}^{-\lambda B/2}$

Converse result: exponential of a bivector $=$ rotor. Form

$$a(\lambda) = \mathrm{e}^{-\lambda B/2}\, a_0\, \mathrm{e}^{\lambda B/2}.$$

Differentiate to get Taylor series:

$$\begin{aligned} a' &= e^{-\lambda B/2}\, a\cdot B\, e^{\lambda B/2} \\ a'' &= e^{-\lambda B/2}\, (a\cdot B)\cdot B\, e^{-\lambda B/2}, \quad \textit{etc.} \end{aligned}$$

NB $a\cdot B$ always a vector, so preserves grade. Get

$$e^{-B/2}\, a\, e^{B/2} = a + a\cdot B + \frac{1}{2!}(a\cdot B)\cdot B + \cdots$$

Lie Algebras and Bivectors

Bivectors are generators of the Lie group, by exponentiation. These generate a Lie algebra. Expresses fact that rotations do not commute. Form compound rotation:

$$Ra\tilde{R} = \tilde{R}_2\tilde{R}_1(R_2R_1a\tilde{R}_1\tilde{R}_2)R_1R_2.$$

The resulting rotor is

$$R = e^{-B/2} = e^{B_2/2}\, e^{B_1/2}\, e^{-B_2/2}\, e^{-B_1/2}$$

Expanding exponentials we find (exercise)

$$B = B_1 \times B_2 + \text{higher order terms}$$

The Baker-Campbell-Hausdorff formula.

Abstract idea. Lie algebra is a linear space (tangent space at the identity element of the group manifold) with a Lie bracket. This is Antisymmetric + Closed + Jacobi identity.

For bivectors, Lie bracket = commutator product, $A \times B$. Gives a third bivector. Jacobi identity is

$$(A \times B) \times C + (C \times A) \times B + (B \times C) \times A = 0$$

Proof Expand out into geometric products. Nothing special about grades. True for any 3 multivectors. One consequence:

$$(a \wedge b) \times B = (a \cdot B) \wedge b - (b \cdot B) \wedge a$$

NB proves closure.

Another view: Basis set of bivectors $\{B_i\}$. Can write

$$B_j \times B_k = C^i_{jk} B_i$$

C^i_{jk} are the structure constants. Compact encoding of properties of a Lie group. Can always construct a matrix rep' of Lie algebra from structure constants

Complex Structures

GA in 2-d gives complex numbers. What about complex *vectors*? Natural idea is work in $2n$-d space. Introduce basis $\{e_i, f_i\}$

$$f_i \cdot f_j = e_i \cdot e_j = \delta_{ij} \qquad e_i \cdot f_j = 0, \quad \forall i, j.$$

Introduce complex structure through doubling bivector

$$J = e_1 f_1 + e_2 f_2 + \cdots + e_n f_n = e_i \wedge f_i$$

Sum of n commuting blades, each an imaginary in own plane. J satisfies

$$J\cdot f_i = (e_j\wedge f_j)\cdot f_i = e_j\delta_{ij} = e_i$$
$$J\cdot e_i = (e_j\wedge f_j)\cdot e_i = -f_i$$

J maps from one half of vector space to other. Follows that

$$J\cdot(J\cdot e_i) = -J\cdot f_i = -e_i, \quad J\cdot(J\cdot f_i) = J\cdot e_i = -f_i,$$

Hence

$$J\cdot(J\cdot a) = (a\cdot J)\cdot J = -a \quad \forall a$$

Phase rotation becomes a rotation in J plane. Expand

$$\begin{aligned} \mathrm{e}^{-J\phi/2}\, a\, \mathrm{e}^{J\phi/2} &= a + \phi\, a\cdot J + \frac{\phi^2}{2!}(a\cdot J)\cdot J + \cdots \\ &= (1 - \frac{\phi^2}{2!} + \cdots)a + (\phi - \frac{\phi^3}{3} - \cdots)a\cdot J \\ &= \cos\phi\, a + \sin\phi\, a\cdot J \end{aligned}$$

HERMITIAN INNER PRODUCT

Complex vectors Z and W:

$$Z_i = x_i + iy_i, \quad \text{and} \quad W_i = u_i + iv_i$$

Hermitian inner product is

$$\langle W|Z\rangle = W_i^* Z_i = u_i x_i + v_i y_i + i(u_i y_i - v_i x_i)$$

Want analog in $2n$-d space. Introduce vectors

$$x = x_i e_i + y_i f_i, \quad \text{and} \quad w = u_i e_i + v_i f_i.$$

Real part of $\langle W|Z\rangle$ is $x\cdot w$. Imaginary part is

$$w\cdot e_i\, x\cdot f_i - w\cdot f_i\, x\cdot e_i = (w\cdot e_i\, x - x\cdot e_i\, w)\cdot f_i$$
$$= [(x\wedge w)\cdot e_i]\cdot f_i = (x\wedge w)\cdot(e_i\wedge f_i) = (x\wedge w)\cdot J.$$

NB Antisymmetric 'bilinear form'. Can now write

$$\langle a|b\rangle = a\cdot b - i\,(a\wedge b)\cdot J$$

Maps from $2n$-d space onto complex numbers.

Unitary Groups

Unitary group is invariance group of Hermitian inner product. Must leave inner product and skew term invariant. Build from rotations. Require that

$$(a'\wedge b')\cdot J = (a\wedge b)\cdot J$$

with $a' = Ra\tilde{R}$, $b' = Rb\tilde{R}$. Find

$$(a'\wedge b')\cdot J = \langle a'b'J\rangle = \langle Ra\tilde{R}Rb\tilde{R}J\rangle$$
$$= \langle ab\tilde{R}JR\rangle = (a\wedge b)\cdot(\tilde{R}JR)$$

Must hold for all a and b, so

$$\tilde{R}JR = J$$

Unitary group U(n) is subgroup of rotor group which leaves J invariant. Get complex groups as sub-groups of real rotation groups! Unusual approach, but has a number of advantages.

With $R = \exp(-B/2)$ must have

$$B \times J = 0.$$

Get bivector form of Lie algebra of the unitary group, u(n). Use Jacobi identity to prove

$$\begin{aligned}[(a\cdot J)\wedge(b\cdot J)]\times J &= -(a\cdot J)\wedge b + (b\cdot J)\wedge a \\ &= -(a\wedge b)\times J.\end{aligned}$$

Follows that

$$[a\wedge b + (a\cdot J)\wedge(b\cdot J)]\times J = 0.$$

Work through all combinations of $\{e_i, f_i\}$. Write down the following Lie algebra basis for u(n):

$$\begin{aligned}E_{ij} &= e_i e_j + f_i f_j \qquad (i < j = 1 \ldots n) \\ F_{ij} &= e_i f_j - f_i e_j \qquad (i < j = 1 \ldots n) \\ J_i &= e_i f_i.\end{aligned}$$

Can establish closure (exercise). Algebra contains J, commutes with all other elements, gives global phase term. Removing this gives special unitary group, SU(n).