January 28, 1999

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 5

SUMMARY

Today will see how GA simplifies and improves our understanding of the important subject of *linear algebra*. This has many applications, and is crucial for the gauge theory of gravity. As a major application we will look in detail at Hamiltonian mechanics and will uncover a geometric framework which forms the natural setting for Hamilton's equations.

- Linear functions of vectors and multivectors.
- The determinant and its geometric meaning.
- Non-orthonormal frames and a selection of useful algebraic identities.
- The adjoints and the inverse.
- Hamilton's equations in a geometric setting.
- Conservation equations.
- Canonical Transformations.

Linear Functions

GA is an index free language. Denote linear functions mapping vectors to vectors as $\mathsf{f}(a)$. Defining property

$$\mathsf{f}(\lambda a + \mu b) = \lambda \mathsf{f}(a) + \mu \mathsf{f}(b).$$

Combine two linear functions f and g, get a third (cf matrix multiplication). Write

$$\mathsf{h}(a) = \mathsf{f}[\mathsf{g}(a)] = \mathsf{fg}(a)$$

Associative so no need for brackets.

Extend action of $\mathsf{f}(a)$ to entire GA by

$$\mathsf{f}(a \wedge b \wedge \cdots \wedge c) = \mathsf{f}(a) \wedge \mathsf{f}(b) \wedge \cdots \wedge \mathsf{f}(c)$$

Right-hand side also a blade with same grade as the original argument. Extended linear functions preserve grade

$$\mathsf{f}(A_r) = \langle \mathsf{f}(A_r) \rangle_r$$

They are also multilinear

$$\mathsf{f}(\lambda A + \mu B) = \lambda \mathsf{f}(A) + \mu \mathsf{f}(B)$$

for any multivectors A and B. This is the way to understand linear algebra!

Example — Rotations. With $\mathsf{R}(a) = Ra\tilde{R}$ have seen extended action to multivectors has same law, so

$$\mathsf{R}(A) = RA\tilde{R} \qquad \forall A$$

Key result. Take a product function $\mathsf{h}(a) = \mathsf{fg}(a)$. See that

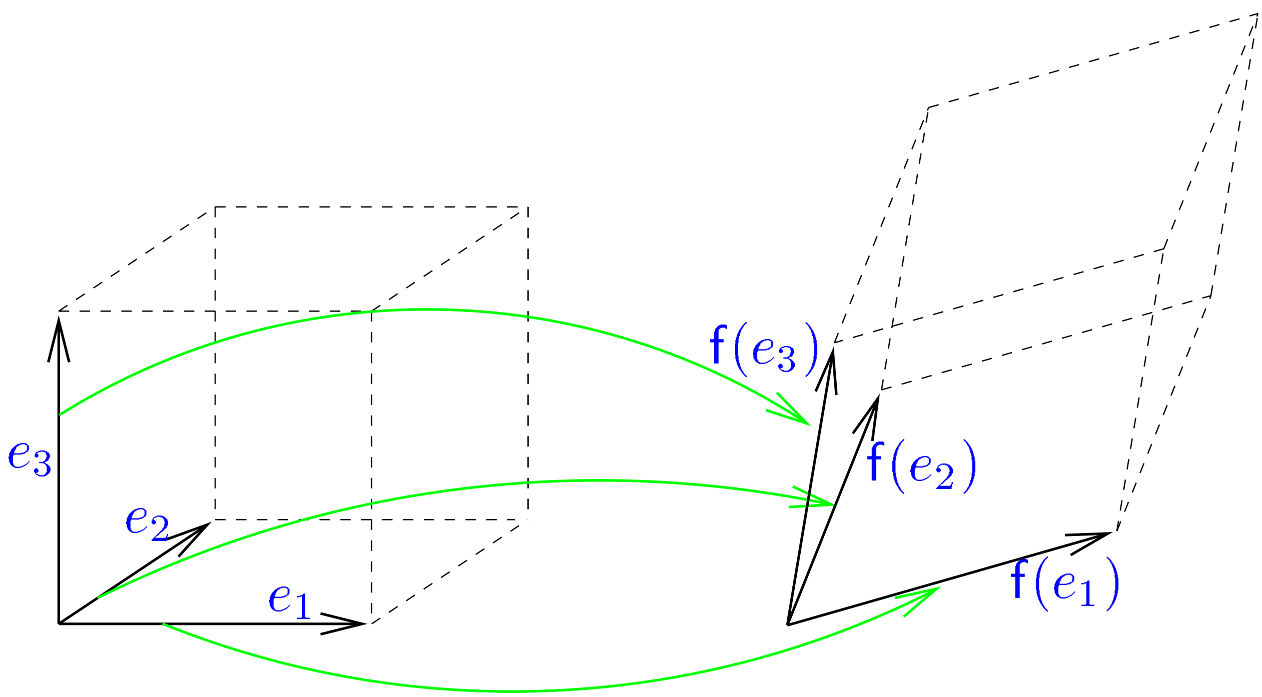
$$\mathsf{h}(a \wedge b \wedge \cdots \wedge c) = \mathsf{fg}(a) \wedge \mathsf{fg}(b) \wedge \cdots \wedge \mathsf{fg}(c)$$
$$= \mathsf{f}[\mathsf{g}(a) \wedge \mathsf{g}(b) \wedge \cdots \wedge \mathsf{g}(c)] = \mathsf{f}[\mathsf{g}(a \wedge b \wedge \cdots \wedge c)]$$

Extension of product = product of extended functions. Still write

$$\mathsf{h}(A) = \mathsf{fg}(A),$$

and right-hand side is unambiguous.

THE DETERMINANT



Unit cube transformed to parallelepiped, sides $\mathsf{f}(e_1)$, $\mathsf{f}(e_2)$ and $\mathsf{f}(e_3)$. Volume is

$$\mathsf{f}(e_1)\wedge\mathsf{f}(e_2)\wedge\mathsf{f}(e_3) = \mathsf{f}(I)$$

Define determinant as the volume scale factor. Linear functions grade preserving for all multivectors. But highest grade element is unique up to scale. Define

$$\mathsf{f}(I) = \det(\mathsf{f})\, I$$

Now prove a key result for determinants. Take $\mathsf{h} = \mathsf{fg}$. Get

$$\mathsf{f}(\mathsf{g}(I)) = \mathsf{f}(\det(\mathsf{g})I) = \det(\mathsf{g})\mathsf{f}(I) = \det(\mathsf{f})\det(\mathsf{g})I$$

Just used multilinearity and extension results. Have proved

$$\det(\mathsf{fg}) = \det(\mathsf{f})\det(\mathsf{g})$$

The simplest proof anywhere!

Non-Orthonormal Frames

Very useful. Unavoidable for special relativity. Take set of n linearly independent vectors $\{e_k\}$. Not necessarily orthogonal. Any vector a decomposes uniquely

$$a = a^k e_k$$

How do we find the components? Need a second set $\{e^k\}$

related to first by

$$e^i \cdot e_j = \delta^i_j$$

The reciprocal frame. With these get

$$e^k \cdot a = e^k \cdot (a^j e_j) = e^k \cdot e_j \, a^j = a^j \delta^k_j = a^k.$$

Note position of indices.

To construct reciprocal frame, see e^1 orthogonal to $\{e_2 \cdots e_n\}$. e^1 perpendicular to hyperplane $e_2 \wedge e_3 \wedge \cdots e_n$. Find by dualisation — multiplication by I. Have

$$e^1 = \alpha \, e_2 \wedge e_3 \wedge \cdots \wedge e_n \, I$$

α found by dotting with e_1

$$1 = e_1 \cdot e^1 = \alpha \, e_1 \wedge e_2 \wedge \cdots \wedge e_n \, I$$

Define

$$E_n = e_1 \wedge e_2 \wedge \cdots \wedge e_n \neq 0$$

so $\alpha = E_n^{-1} I^{-1}$. Arrive at useful formula

$$e^k = (-1)^{k+1} e_1 \wedge \ldots \wedge \check{e}_k \wedge \cdots \wedge e_n E_n^{-1}$$

$\check{e}_k$ term missing from product. Purely geometric reasoning led quickly to an algebraic formula. Can be directly applied.

Will use arbitrary frames and reciprocals where frames needed.

Some Useful Results

Basic identity

$$a = a^k e_k = a \cdot e^k e_k = a \cdot e_k \, e^k$$

Build up useful results. First

$$\begin{aligned} e_k \, e^k \cdot (a \wedge b) &= e_k (e^k \cdot a \, b - e^k \cdot b \, a) \\ &= ab - ba = 2 a \wedge b \end{aligned}$$

Extends inductively to

$$e_k \, e^k \cdot A_r = r A_r$$

for grade-r multivector. Next use

$$e^k e_k = e^k (e_j \cdot e_k \, e^j) = e_j \cdot e_k e^k e^j$$

$e_j \cdot e_k$ symmetric on $j, k \Rightarrow$ only get scalar contribution

$$e_k e^k = e_k \cdot e^k = n$$

where n is dimension of vector space. Follows that

$$e_k \, e^k \wedge A_r = e_k (e^k A_r - a^k \cdot A_r) = (n - r) A_r$$

Combining above gives

$$\begin{aligned} e_k A_r e^k &= (-1)^r e_k (e^k \wedge A_r - e^k \cdot A_r) \\ &= (-1)^r (n - 2r) A_r \end{aligned}$$

Recovering a Rotor

Two arbitrary non-orthonormal frames $\{e_k\}$ and $\{f_k\}$ related by a rotation,

$$f_k = R e_k \tilde{R}$$

How do we find R? Work in 3-d, so

$$\tilde{R} = \mathrm{e}^{B/2} = \cos(|B|/2) + \sin(|B|/2) B/|B|$$

Find that

$$\begin{aligned} e_k \tilde{R} e^k &= 3\cos(|B|/2) - \sin(|B|/2) B/|B| \\ &= 4\cos(|B|/2) - \tilde{R}. \end{aligned}$$

Now form

$$f_k e^k = R e_k \tilde{R} e^k = 4\cos(|B|/2) R - 1.$$

R is scalar multiple of $1 + f_k e^k$, so

$$R = \frac{1 + f_k e^k}{|1 + f_k e^k|} = \frac{\psi}{\sqrt{(\psi\tilde{\psi})}}$$

where $\psi = 1 + f_k e^k$. Recovers the R directly from frame vectors.

The Adjoint

The reverse map $\bar{\mathsf{f}}(a)$

$$a\cdot\mathsf{f}(b) = \bar{\mathsf{f}}(a)\cdot b, \qquad \forall a, b$$

Decomposing $\bar{\mathsf{f}}(a)$ in a frame

$$\bar{\mathsf{f}}(a) = \bar{\mathsf{f}}(a)\cdot e_k\, e^k = a\cdot\mathsf{f}(e_k)e^k$$

Same as transpose of a matrix/tensor.

Construct extension

$$\begin{aligned}\bar{\mathsf{f}}(a\wedge b) &= e_i\wedge e_j\, a\cdot\mathsf{f}(e^i)\, b\cdot\mathsf{f}(e^j) \\ &= \tfrac{1}{2}e_i\wedge e_j\,(a\wedge b)\cdot\mathsf{f}(e^j\wedge e^i)\end{aligned}$$

Extension of adjoint $=$ *adjoint of extension*. Write

$$A_r\cdot\bar{\mathsf{f}}(B_r) = \mathsf{f}(A_r)\cdot B_r$$

Extend to mixed grades, *e.g.*

$$\begin{aligned}a\cdot\mathsf{f}(b\wedge c) &= a\cdot\mathsf{f}(b)\,\mathsf{f}(c) - a\cdot\mathsf{f}(c)\,\mathsf{f}(b) \\ &= \mathsf{f}(\bar{\mathsf{f}}(a)\cdot b\, c - \bar{\mathsf{f}}(a)\cdot c\, b) = \mathsf{f}[\bar{\mathsf{f}}(a)\cdot(b\wedge c)]\end{aligned}$$

Similar argument, get remarkably useful formulae

$$\begin{aligned}\mathsf{f}(A_r)\cdot B_s &= \mathsf{f}[A_r\cdot\bar{\mathsf{f}}(B_s)] \qquad && r \geq s \\ A_r\cdot\bar{\mathsf{f}}(B_s) &= \bar{\mathsf{f}}[\mathsf{f}(A_r)\cdot B_s] \qquad && r \leq s\end{aligned}$$

The Inverse

Preceding formulae quickly yield the inverse function! Set $B_s = I$ in second formula,

$$A_r \det(\mathsf{f}) I = \bar{\mathsf{f}}[\mathsf{f}(A_R) I]$$

Write as

$$A_r = \bar{\mathsf{f}}[\mathsf{f}(A_r) I] I^{-1} \det(\mathsf{f})^{-1}$$

The green terms undo effect f. Must represent the inverse function. Therefore have

$$\begin{aligned} \mathsf{f}^{-1}(A) &= \det(\mathsf{f})^{-1} I \bar{\mathsf{f}}(I^{-1} A) \\ \bar{\mathsf{f}}^{-1}(A) &= \det(\mathsf{f})^{-1} I \mathsf{f}(I^{-1} A) \end{aligned}$$

No simpler proof anywhere else! And very useful, can be coded in symbolic algebra packages (Maple).

Example - Rotations

Rotation $\mathsf{R}(a) = Ra\tilde{R}$. Adjoint $\bar{\mathsf{R}}(a)$ found from

$$e_k \, a \cdot \mathsf{R}(e^k) = e_k \langle a R e^k \tilde{R} \rangle = e_k \, e^k \cdot (\tilde{R} a R) = \tilde{R} a R$$

Extends to $\bar{\mathsf{R}}(A) = \tilde{R} A R$. Inverse given by

$$\mathsf{R}^{-1}(A) = \det(\mathsf{R})^{-1} I \tilde{R} I^{-1} A R = \tilde{R} A R$$

as $\det(\mathsf{R}) = 1$. Inverse = adjoint — an orthonormal transformation.

Hamiltonian Mechanics

Possess necessary ideas to geometrise Hamiltonian dynamics. Start with Lagrangian $L(q_i, \dot{q}_i, t)$, $\{q_i\}$ are n arbitrary coordinates. Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}.$$

Equivalent to Hamilton's equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Hamiltonian $H(q_i, p_i, t)$ given by

$$H(q_i, p_i, t) = p_i \dot{q}_i - L(q_i, \dot{q}_i, t)$$

with $\dot{q}_i$ expressed in terms of the p_i

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

n 2nd order equations $\mapsto$ $2n$ first order equations.

Natural setting is $2n$-d 'doubled' space $\{e_i, f_i\}$. Define point in phase space by the vector

$$x = p_i e_i + q_i f_i$$

Hamiltonian is function of this vector, $H = H(x,t)$, so that

$$\nabla H = e_i \frac{\partial H}{\partial p_i} + f_i \frac{\partial H}{\partial q_i} = \dot{q}_i e_i - \dot{p}_i f_i$$

where ∇ is gradient operator

$$\nabla = e_i \frac{\partial}{\partial p_i} + f_i \frac{\partial}{\partial q_i}$$

Hamilton's equations specify a phase space trajectory $x(t)$

$$\begin{aligned} \dot{x} &= \dot{p}_i e_i + \dot{q}_i f_i = -\tfrac{\partial H}{\partial q_i} e_i + \tfrac{\partial H}{\partial p_i} f_i \\ &= (\tfrac{\partial H}{\partial q_i} f_i) \cdot (e_j \wedge f_j) + (\tfrac{\partial H}{\partial p_i} e_i) \cdot (e_j \wedge f_j) \end{aligned}$$

Recover the bivector J! Hamilton's equations now become

$$\dot{x} = \nabla H \cdot J$$

Number of advantages

1. Easy to prove *e.g.* conservation theorems and Liouville's theorem.
2. Canonical transformations understood geometrically.
3. Poisson bracket naturally incorporated (later).
4. Extends to more complicated systems. Phase space $\mapsto$ manifold. $J \mapsto$ symplectic bivector. Equation structure unchanged.
5. Natural setting for instability and chaos.

Derivatives and Flows

Introduce $2n$-d fixed frame $\{e_k\}$. Write

$$x = x^k e_k, \qquad x^k = e^k \cdot x$$

Have

$$\nabla = e^k \frac{\partial}{\partial x^k} = e^k \, e_k \cdot \nabla$$

Follows from chain rule that for functions of x only

$$\frac{d}{dt} = \frac{\partial x^k}{\partial t} \frac{\partial}{\partial x^k} = \frac{\partial x}{\partial t} \cdot e^k \, e_k \cdot \nabla = \dot{x} \cdot \nabla$$

Take scalar function $f(x)$ on phase space. (Independent of t.) Evolution along phase space trajectory $x(t)$ determined by

$$\dot{f} = \dot{x} \cdot \nabla f = (\nabla f \wedge \nabla H) \cdot J$$

Immediately get $\dot{H} = 0$. If H invariant along constant direction a in phase space, $a \cdot \nabla \, H = 0$, get

$$-[(a \cdot J) \cdot J] \cdot \nabla H = (a' \wedge \nabla H) \cdot J = 0$$

where $a' = a \cdot J$. From above see that

$$\frac{d}{dt}(a' \cdot x) = 0 = \frac{d}{dt}[(x \wedge a) \cdot J],$$

so $(x \wedge a) \cdot J$ is conserved quantity.

CANONICAL TRANSFORMATIONS

Equation $\dot{x} = \nabla H \cdot J$ is geometric. Can decompose in any coordinate frame. Gives passive transformations. Useful, but not the whole story. Suppose have different set of coordinates and canonical momenta P_i, Q_i. Form a different vector

$$x' = P_i e_i + Q_i f_i.$$

The $\{P_i, Q_i\}$ are functions of original $\{p_i, q_i\}$. View new vector x' as function of the old, x. Write

$$x' = f(x).$$

This is an active transformation — a displacement. Points actually moved around in phase space. No restriction on form of $f(x)$ other than invertible.

Assume $f(x)$ independent of t. Form

$$\frac{d}{dt}x' = \frac{d}{dt}f(x) = \dot{x}\cdot\nabla f(x)$$

Define *differential* of $f(x)$

$$\mathsf{f}(a) = a\cdot\nabla f(x)$$

A linear function of a. Also position dependent. Sometimes write $\mathsf{f}(a) = \mathsf{f}(a\,; x)$ but suppress the x where possible.

Now have

$$\frac{d}{dt}x' = \dot{x}\cdot\nabla f(x) = \mathsf{f}(\dot{x})$$

Next relate gradients with respect to x and x'. Have

$$x' = x^{k'}e_k, \quad x^{k'} = e^k\cdot x', \quad \nabla' = e^k\frac{\partial}{\partial x^{k'}} = e^k e_k\cdot\nabla'$$

Find that

$$\begin{aligned}\nabla &= e^k\, e_k\cdot\nabla = e^k\, e_k\cdot\nabla x^{j'}\tfrac{\partial}{\partial x^{j'}} \\ &= e^k\,(e_k\cdot\nabla x')\cdot\nabla' = e^k\,\mathsf{f}(e_k)\cdot\nabla' = \bar{\mathsf{f}}(\nabla')\end{aligned}$$

So $\nabla = \bar{\mathsf{f}}(\nabla')$. Very neat again! Now get

$$\frac{d}{dt}x' = \mathsf{f}(\dot{x}) = \mathsf{f}[\nabla H\cdot J] = \bar{\mathsf{f}}^{-1}(\nabla H)\cdot\mathsf{f}(J).$$

But transformed Hamiltonian is $H'(x') = H(x)$, so

$$\bar{\mathsf{f}}^{-1}(\nabla H) = \nabla' H(x) = \nabla' H'$$

Equations of motion after transformation are now

$$\frac{d}{dt}x' = (\nabla' H')\cdot\mathsf{f}(J)$$

Will still be Hamiltonian in form if

$$\mathsf{f}(J) = J.$$

Defines a canonical transformation. $\mathsf{f}(a)$ is a symplectic transformation. Examples include unitary transformations.