February 2, 1999

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 6

SUMMARY

1. General linear transformations.

- The Balanced Algebra $\mathcal{G}_{n,n}$.
- The doubling bivector in $\mathcal{G}_{n,n}$ and null vectors.
- General linear transformations and the singular value decomposition.
- Linear transformations with rotors.

2. Projective geometry.

- A different use for geometric algebra — points as vectors.
- The join and meet operations and a new use for duality.
- Desargues' Theorem.
- Homogeneous coordinates and projective bivectors.
- Invariants and computer vision.

THE BALANCED ALGEBRA $\mathcal{G}_{n,n}$

Start with an n-dimensional orthonormal basis $\{e_k\}$, $e_i \cdot e_j = \delta_{ij}$. Introduce a second frame $\{f_k\}$:

$$f_i \cdot f_j = -\delta_{ij}, \qquad e_i \cdot f_j = 0.$$

Algebra generated by equal 'balanced' numbers of vectors with positive and negative square. Labeled $\mathcal{G}_{n,n}$. Introduce the balanced 'doubling' bivector

$$K = e_i f_i = e_1 \wedge f_1 + e_2 \wedge f_2 + \cdots + e_n \wedge f_n.$$

Properties:

$$e_i \cdot K = f_i, \qquad f_i \cdot K = -f_i \cdot f_j \, e_j = e_i$$

$$\Rightarrow \quad (a \cdot K) \cdot K = K \cdot (K \cdot a) = a \quad \forall a$$

Crucial sign difference cf. complex bivector J. Generates null structure. Take vector a in $\mathcal{G}_{n,n}$. Define

$$n = a \pm a \cdot K$$

Get

$$n^2 = a^2 \pm 2a \cdot (a \cdot K) + (a \cdot K)^2 = a^2 - \langle a \cdot K \, K a \rangle$$
$$= a^2 - [(a \cdot K) \cdot K] \cdot a = a^2 - a^2 = 0$$

n is a null vector — important in relativity. K splits $\mathcal{G}^1_{n,n}$ into two separate null spaces,

$$a_+ = \frac{1}{2}(a + a\cdot K), \qquad a_- = \frac{1}{2}(a - a\cdot K)$$

Use $\mathcal{V}_n$ for space of vectors a_+. Vectors characterised by

$$a_+\cdot K = a_+ \quad \forall a_+ \in \mathcal{V}_n.$$

All vectors in $\mathcal{V}_n$ square to zero — form a Grassmann algebra. (Quantum field theory and supersymmetry.)

THEOREM

Every non-singular linear function $a \mapsto \mathsf{f}(a)$*, can be represented in* $\mathcal{V}_n$ *by a transformation*

$$a_+ \mapsto M a_+ M^{-1},$$

where M *is a geometric product of an even number of unit vectors.*

Vector a in $\mathcal{G}_n \mapsto$ null vector a_+ in $\mathcal{G}_{n,n}$. Acted on by M such that

$$\mathsf{f}(a) + \mathsf{f}(a)\cdot K = M(a + a\cdot K)M^{-1}.$$

Defines a 'double-cover' map between linear functions $\mathsf{f}(a)$ and multivectors $M \in \mathcal{G}_{n,n}$. Must remain in $\mathcal{V}_n$, so

$$(Ma_+M^{-1})\cdot K = Ma_+M^{-1}$$

so have

$$a_+ = M^{-1}(Ma_+M^{-1})\cdot K\,M = a_+\cdot(M^{-1}KM)$$

We require $M^{-1}KM = K$, or

$$MK = KM$$

LIE ALGEBRA

With M a product of an even number of unit vectors have $M\tilde{M} = \pm 1$. Subgroup with $M\tilde{M} = 1$ are rotors in $\mathcal{G}_{n,n}$. Generators are bivectors which commute with K. Form

$$\begin{aligned} [(a\cdot K)\wedge(b\cdot K)]\times K &= a\wedge(b\cdot K) + (a\cdot K)\wedge b \\ &= (a\wedge b)\times K \end{aligned}$$

So have

$$[a\wedge b - (a\cdot K)\wedge(b\cdot K)]\times K = 0$$

Run through all combinations of $\{e_i, f_i\}$. Produce basis

$$\begin{aligned} E_{ij} &= e_ie_j - f_if_j \qquad (i < j = 1\ldots n) \\ F_{ij} &= e_if_j - f_ie_j \qquad (i < j = 1\ldots n) \\ K_i &= e_if_i. \end{aligned}$$

Cf. unitary group! Only difference due to signature of underlying space.

Singular Value Decomposition

From non-singular function $\mathsf{f}(a)$ form symmetric function $\bar{\mathsf{f}}\mathsf{f}(a)$. Has a spectrum of orthonormal eigenvectors d_i and eigenvalues λ_i,

$$\bar{\mathsf{f}}\mathsf{f}(d_i) = \lambda_i d_i$$

(No sum here.) The λ_i are positive:

$$d_i \cdot \bar{\mathsf{f}}\mathsf{f}(d_i) = [\mathsf{f}(d_i)]^2 = \lambda_i (d_i)^2$$

Only works for Euclidean spaces. Write

$$\bar{\mathsf{f}}\mathsf{f}(a) = \sum_k \lambda_k \, a \cdot d_k \, d_k$$

and take square root

$$\mathsf{d}(a) = \sum_k (\lambda_k)^{1/2} a \cdot d_k \, d_k$$

Define $\mathsf{S} = \mathsf{f}\mathsf{d}^{-1}$. Satisfies

$$\bar{\mathsf{S}}\mathsf{S} = \bar{\mathsf{d}}^{-1}\bar{\mathsf{f}}\mathsf{f}\mathsf{d}^{-1} = \bar{\mathsf{d}}^{-1}\mathsf{d}^2\mathsf{d}^{-1} = \mathsf{I},$$

so is an orthonormal transformation. (I is the identity.) Have

$$\mathsf{f}(a) = \mathsf{S}\mathsf{d}(a),$$

Linear function $=$ product of a series of dilations (a symmetric function) followed by orthonormal transformation.

For matrices can write $\boldsymbol{M} = \boldsymbol{S\Lambda R}$. $\boldsymbol{S}, \boldsymbol{R}$ are orthonormal matrices and $\boldsymbol{\Lambda}$ is diagonal. The SVD. Very useful — important in signal processing and data analysis.

Proof of Theorem

1. Rotations.

Generated by E_{ij}. Jointly rotate the $\{e_i\}$ and $\{f_i\}$ by same amount.

2. Reflections.

Reflection in $\mathcal{G}_n$ generated by unit vector n. Define

$$\hat{n} = n \cdot K, \qquad \hat{n}^2 = -1.$$

Take $M = n\hat{n}$. Commutes with K:

$$\begin{aligned} n\hat{n}K &= 2n\,\hat{n}\cdot K + nK\hat{n} \\ &= 2(n^2 + \hat{n}^2) + Kn\hat{n} = Kn\hat{n} \end{aligned}$$

Action on vector $a_+ \in \mathcal{V}_n$:

$$\begin{aligned} -n\hat{n}a_+\hat{n}n &= -n\hat{n}a\hat{n}n - (n\hat{n}a\hat{n}n)\cdot K \\ &= -nan - (nan)\cdot K \end{aligned}$$

since $a\cdot\hat{n} = 0$. This is required action. Have $(n\hat{n})^2 = +1$ so not a rotor. Only need these for reflections.

3. Dilations

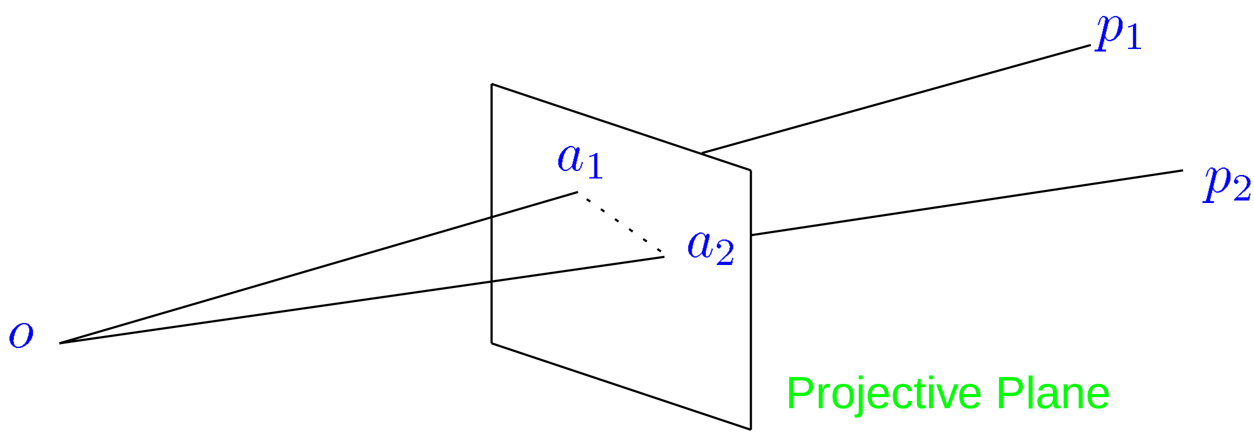
Take bivector $n\hat{n}$ again — constructed from F_{ij} and K_i. Action of rotor on $n_+ = n + \hat{n}$:

$$
\begin{aligned}
\mathrm{e}^{-\lambda n\hat{n}/2}\, n_+\, \mathrm{e}^{\lambda n\hat{n}/2} &= \mathrm{e}^{-\lambda n\hat{n}}\, n_+ \\
&= [\cosh(\lambda) - n\hat{n}\sinh(\lambda)](n+\hat{n}) \\
&= [\cosh(\lambda) + \sinh(\lambda)](n+\hat{n}) = \mathrm{e}^{\lambda}\, n_+
\end{aligned}
$$

A dilation. Vectors perpendicular to n have image in $\mathcal{V}_n$ which commutes with $n\hat{n}$, so unaffected. Completes the proof.

All Lie algebras can be realised as bivector algebras. All matrix operators as geometric products of even number of unit vectors. Can simplify many proofs in linear algebra like this. Yet to be fully exploited.

Projective Geometry



Vectors in 3-d space projected onto a 2-d plane. Points in the plane (a_1, a_2) represented by vectors is a space of one dimension higher. Magnitude of the vector is unimportant — a and λa represent same point. Does not mean there is no role for the dot product.

Line joining a_1, a_2 is result of projecting the $a_1 \wedge a_2$ plane onto the projective plane. Define the join of the points a, b by

$$\text{join}(a, b) = a \wedge b$$

Bivectors used to represent lines now. Keep taking exterior products to define (projectively) higher dimensional objects. Get condition that a, b, c lie on a line:

$$a \wedge b \wedge c = 0$$

For 3-d problems we need to be in 4-d space. Have 6 independent bivectors. Lines described by blades, so $B \wedge B = 0$. Commuting blades = non-intersecting lines.

To find intersection of lines, *etc*. use duality

$$A_r^* = A_r I = A_r \cdot I = (-1)^{r(n-r)} I A$$

where I is the pseudoscalar. In 3-d the dual of a line (a bivector) is a conjugate point (a vector). I interchanges inner and outer products

$$A_r \cdot (B_s I) = A_r \wedge B_s \, I \qquad r + s \leq n$$
$$A_r \wedge (B_s I) = A_r \cdot B_s \, I \qquad r \leq s.$$

I any pseudoscalar which spans space of all vectors contained in A_r and B_s.

Define the meet $A \vee B$ by a 'de Morgan rule'

$$(A \vee B)^* = A^* \wedge B^*$$

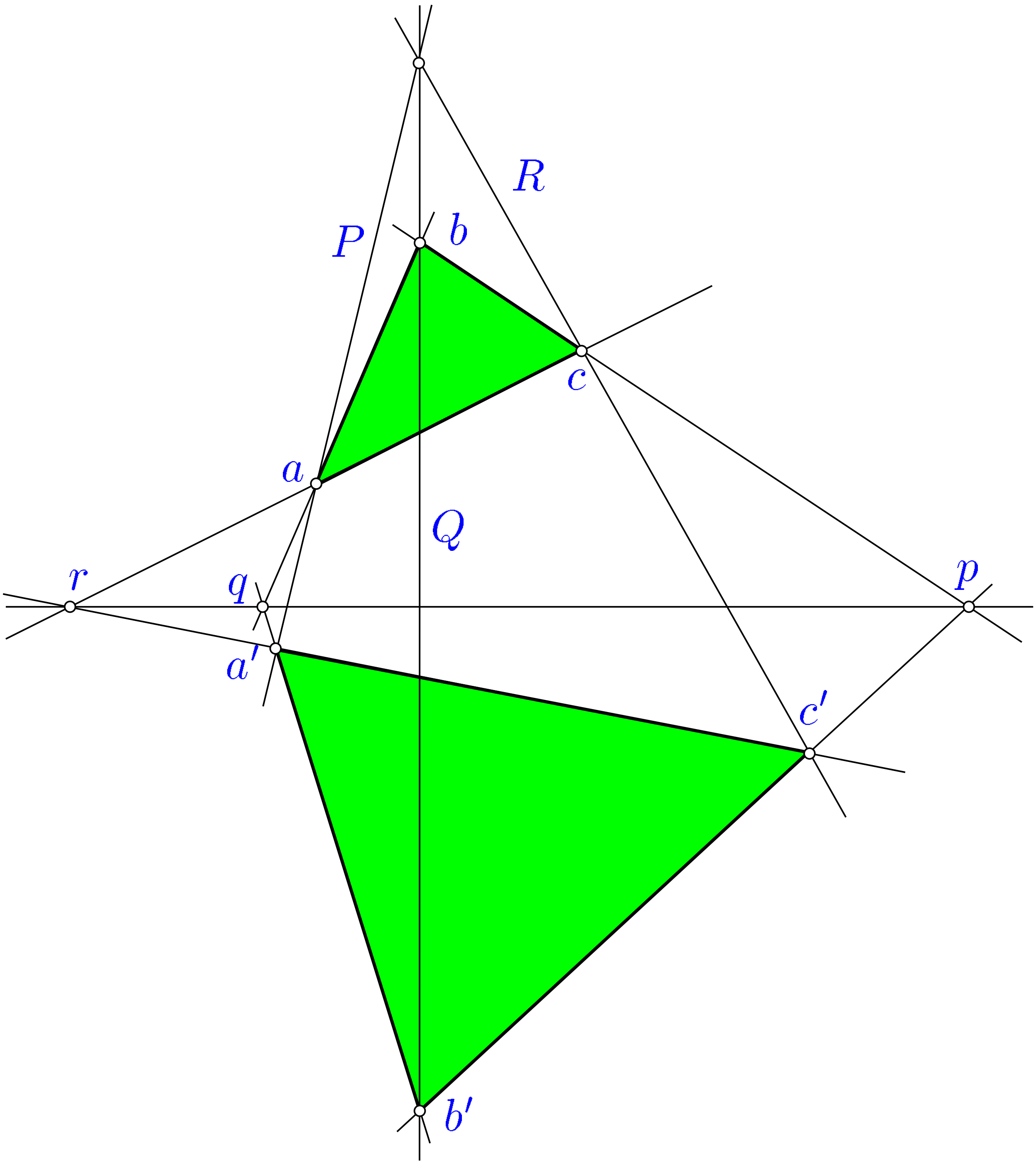
with dual formed with respect to pseudoscalar of space from vectors in blades A and B.

Example — two lines in a plane. I has grade 3, work in $\mathcal{G}_3$. The dual of the meet = join of two vectors (a bivector). Meet is a vector — 2 lines meet at a point! Have

$$A \vee B = (A^* \wedge B^*) I^{-1} = A \times B \, I$$

A and B are bivectors in $\mathcal{G}_3$.

DESARGUES' THEOREM



Two triangles in a plane, points a, b, c and a', b', c'. The lines P, Q, R meet at a point if and only if the points p, q, r all lie on a line. The triangles are then projectively related.

DESARGUES' THEOREM

Have lines

$$A = b \wedge c, \quad B = c \wedge a, \quad C = a \wedge b$$

with same for A', B', C' in terms of a', b', c'. Two sets of points determine the lines

$$P = a \wedge a', \quad Q = b \wedge b', \quad R = c \wedge c'$$

Two sets of lines determine the points

$$p = A \times A' \, I, \quad q = B \times B' \, I, \quad r = C \times C' \, I.$$

Three lines P, Q, R meet at a point:

$$(P \vee Q) \wedge R = \langle P \times Q \, RI \rangle_3 = \langle PQR \rangle I$$

Three points p, q, r fall on a line:

$$\begin{aligned} p \wedge q \wedge r &= \langle A \times A' \, I \, B \times B' \, I \, C \times C' \, I \rangle_3 \\ &= -I \langle A \times A' \, B \times B' \, C \times C' \rangle. \end{aligned}$$

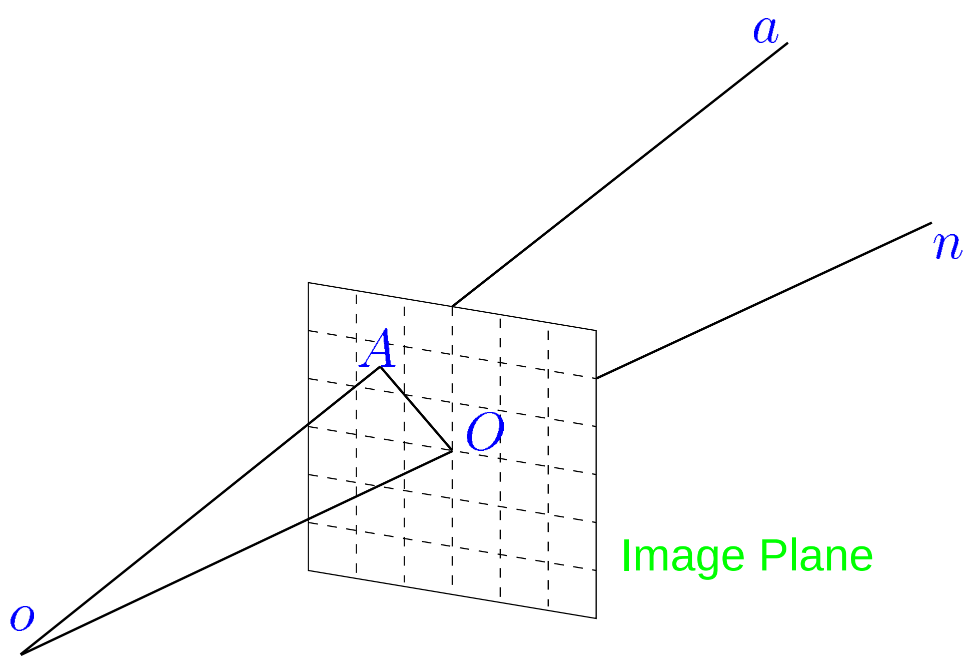
Theorem is proved by the algebraic identity (exercise)

$$JJ' \langle a \wedge a' \, b \wedge b' \, c \wedge c' \rangle = \langle A \times A' \, B \times B' \, C \times C' \rangle$$

where

$$J = a \wedge b \wedge c, \qquad J' = a' \wedge b' \wedge c'$$

Homogeneous Coordinates



Want relationship between coordinates in image plane and 3-d vector. OA in a 2-d space. Relate to 3-d algebra by

$$n + OA = \lambda a$$

Choose scale with $n^2 = 1$, $\Rightarrow \lambda = (a \cdot n)^{-1}$,

$$OA = \frac{a - a \cdot n n}{a \cdot n} = \frac{a \wedge n}{a \cdot n} n$$

Represent line OA in 2-d with the bivector

$$A = \frac{a \wedge n}{a \cdot n}$$

Projective map between bivectors and vectors in a space one dimension lower. Introduce coordinate frame with $n = e_3$.

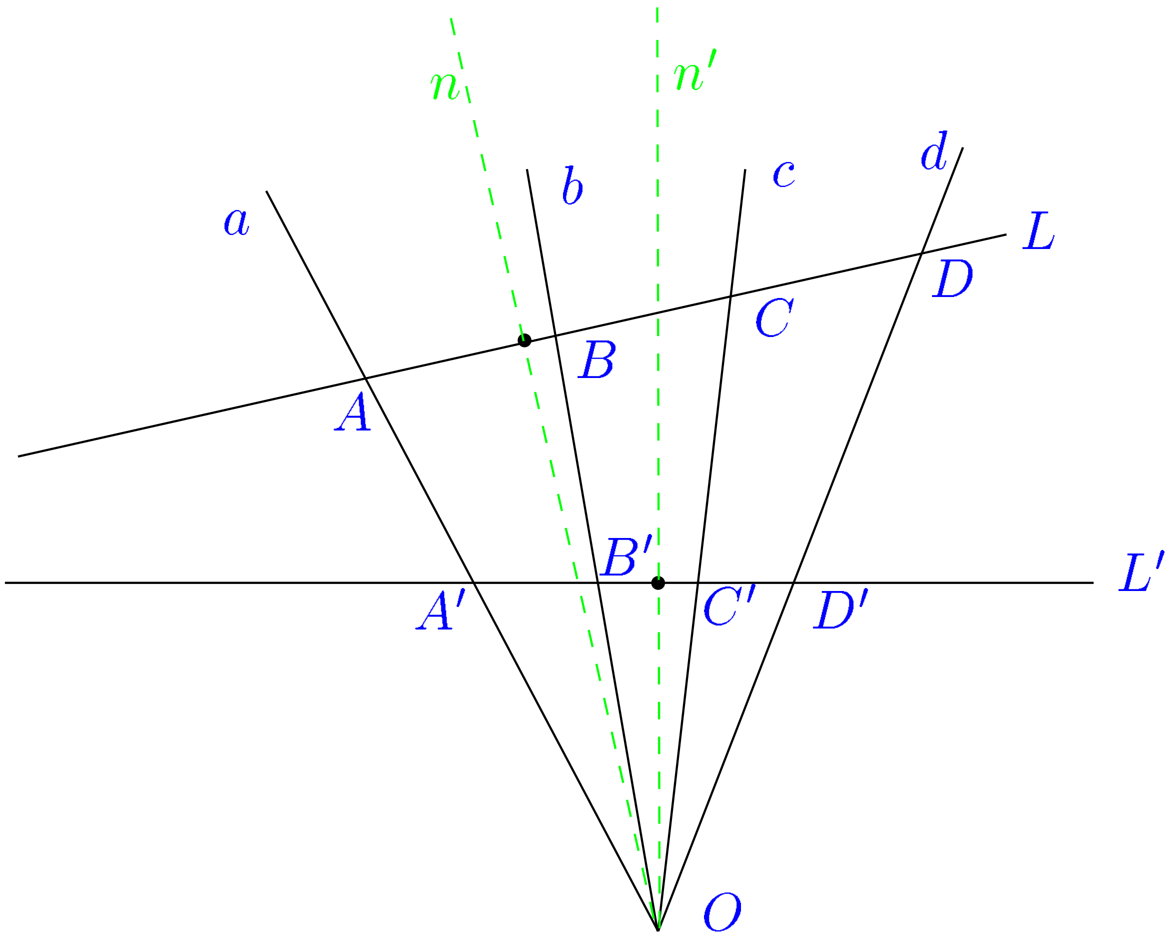
Get

$$A = \frac{a_1}{a_3} e_1 e_3 + \frac{a_2}{a_3} e_2 e_3 = A_1 E_1 + A_2 E_2$$

Components $A_i = a_i/a_3$ are homogeneous coordinates — independent of scale. Often measure these. Map between a and A is nonlinear in 3-d. Turns into a linear map by representing points in 3-d with vectors in 4-d!

NB. $\{E_1, E_2\}$ have negative square. Can get round this.

INVARIANTS



Want to find projective invariants — independent of camera position. Use these to check point matches. Consider 1-d example. Lines defined by a, b, c, d project out two sets of points on two different lines. With n unit normal to the line, have

$$OA = A = \frac{a\wedge n}{a\cdot n}$$

Invariants formed from ratios of lengths. Form bivector for AB,

$$AB = \frac{b\wedge n}{b\cdot n} - \frac{a\wedge n}{a\cdot n} = \frac{(a\cdot n\, b - b\cdot n\, a)\wedge n}{a\cdot n\, b\cdot b}$$

$$= \frac{[(b\wedge a)\cdot n]\wedge n}{a\cdot n\, b\cdot b} = \frac{b\wedge a}{a\cdot n b\cdot n}$$

Need combination which is independent of n. Form

$$\rho = \frac{AC}{BC}\frac{BD}{AD} = \frac{a\wedge c}{b\wedge c}\frac{b\wedge d}{a\wedge d}$$

Manifestly independent of the chosen projection.

For projection of 3-d image onto 2-d camera plane, invariant formed from ratios of trivectors. These are areas in the camera plane. With 5 point matches, vectors $a_1 \cdots a_5$ produce 5 projected points $A_1 \cdots A_5$. Invariant formed by

$$\frac{a_5\wedge a_4\wedge a_3}{a_5\wedge a_1\wedge a_3}\frac{a_5\wedge a_2\wedge a_1}{a_5\wedge a_2\wedge a_4} = \frac{A_{543}}{A_{513}}\frac{A_{521}}{A_{524}}$$

where $A_{ijk} =$ projected area of triangle with vertices A_i, A_j, A_k. Again, easy to prove projective invariance.