February 4, 1999

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 7

SUMMARY

In this lecture we will look at what GA has to tell us about the subject of vector calculus, and how the geometric product provides a first order, invertible vector derivative.

- The vector derivative. Combining derivatives with GA to form a geometric calculus.
- Curvilinear coordinates, coordinate frames and frame-free linear algebra.
- Geometric Calculus in the plane and the Cauchy-Riemann equations.
- The fundamental theorem of calculus. Relating surface and volume integrals.
- Analytic functions and the Cauchy integral formula

THE VECTOR DERIVATIVE

Points represented by vector x. Fixed frame $\{e^k\}$, coordinates $x^k = e^k \cdot x$. Define

$$\nabla = \sum_k e^k \frac{\partial}{\partial x^k}$$

Dot ∇ with a, get directional derivative in a direction

$$a \cdot \nabla F(x) = \lim_{\epsilon \mapsto 0} \frac{F(x+\epsilon a) - F(x)}{\epsilon}$$

$F(x)$ a multivector-valued function of position.

Scalar field $\phi(x)$ — $\nabla \phi$ returns the gradient (vector pointing in direction of steepest increase).

Vector field $J(x)$ — Can form geometric product ∇J.

- Scalar part:

$$\nabla \cdot J = \frac{\partial J^k}{\partial x^k} = \partial_k J^k$$

  The divergence. Have written

$$\partial_i = \frac{\partial}{\partial x^i}$$

- Bivector part:

$$\nabla \wedge J = e^i \wedge (\partial_i J) = e^i \wedge e^j \, \partial_i J_j$$

  Antisymmetrised terms in $\partial_i J_j$. In 3-d get components of the *curl*.

$$\nabla \wedge J = I \nabla \times J$$

A bivector, of course!

MULTIVECTOR FIELDS

Definitions extend simply to multivector fields

$$\nabla A = e^k \partial_k A$$

and define divergence and curl by

$$\nabla \cdot A_r = \langle \nabla A_r \rangle_{r-1}, \qquad \nabla \wedge A_r = \langle \nabla A_r \rangle_{r+1}$$

The curl of a curl vanishes because partial derivatives commute

$$\begin{aligned} \nabla \wedge (\nabla \wedge A) &= e^i \wedge \partial_i (e^j \wedge \partial_j A) \\ &= e^i \wedge e^j \, \partial_i \partial_j A = 0 \end{aligned}$$

Get dual result, the divergence of a divergence vanishes

$$\nabla \cdot (\nabla \cdot A) = 0.$$

∇ does not commute with multivectors. Need conventions for scope of operator:

1. In absence of brackets, ∇ acts to immediate right.
2. ∇ acts on all of the terms in adjacent bracket.

3. When ∇ acts on non-adjacent multivector use overdots for scope.

$$\dot{\nabla} A \dot{B} = e^k A \partial_k B$$

So A not differentiated. Can write Leibniz' rule

$$\nabla(AB) = \nabla AB + \dot{\nabla} A \dot{B}$$

Also use for linear functions

$$\dot{\nabla}\dot{\mathsf{f}}(a) = \nabla \mathsf{f}(a) - e^k \mathsf{f}(\partial_k a)$$

Linear Algebra

Basic relation

$$\nabla(x \cdot a) = e^i \partial_i (x^j e_j) \cdot a = e^i\, e_j \cdot a\, \delta_i^j = e^i\, e_i \cdot a = a$$

Replace frame contractions by derivatives. Vector variable a derivative ∂_a. Have

$$\begin{aligned} \partial_a\, a \cdot A_r &= r A_r \\ \partial_a\, a \wedge A_r &= (n-r) A_r \end{aligned}$$

Similarly, write trace of a linear function

$$\mathrm{Tr}(\mathsf{f}) = \partial_a \cdot \mathsf{f}(a)$$

Removes frames, emphasises geometric content.

Curvilinear Coordinates

Often need non-Cartesian coordinate systems. Set of scalar functions $\{x^i(x)\}$. Express $F(x)$ as $F(x^i)$. Chain rule gives

$$\nabla F = \nabla x^i \, \partial_i F = e^i \, \partial_i F$$

Defines (contravariant) frame vectors $\{e^i\}$

$$e^i = \nabla x^i$$

In Euclidean spaces perpendicular to surfaces of constant x^i. Zero curl:

$$\nabla \wedge e^i = \nabla \wedge (\nabla x^i) = 0$$

Reciprocal frame from (covariant) coordinate vectors

$$e_i = \partial_i x$$

Direction of increasing x^i coordinate, others fixed. Reciprocal because

$$e_i \cdot e^j = (\partial_i x) \cdot \nabla x^j = \partial_i x^j = \delta_i^j .$$

Work with both. Avoid 'weighting factors' for orthogonal frames

$$e_i = h_i \hat{e}_i, \quad e^i = h_i^{-1} \hat{e}_i .$$

Particularly bad if signature not Euclidean.

2-D GEOMETRIC CALCULUS

Write the vector x in right-handed orthonormal frame

$$x = x^1 e_1 + x^2 e_2 = \mathrm{x} e_1 + \mathrm{y} e_2$$

(NB different fonts.) Vector derivative is

$$\nabla = e_1 \partial_{\mathrm{x}} + e_2 \partial_{\mathrm{y}} = e_1 (\partial_{\mathrm{x}} + I \partial_{\mathrm{y}})$$

with $I = e_1 e_2$. Act on vector $a = u e_1 - v e_2$:

$$\nabla a = (e_1 \partial_{\mathrm{x}} + e_2 \partial_{\mathrm{y}})(u e_1 - v e_2)$$
$$= \frac{\partial u}{\partial \mathrm{x}} - \frac{\partial v}{\partial \mathrm{y}} - I \left(\frac{\partial v}{\partial \mathrm{x}} + \frac{\partial u}{\partial \mathrm{y}} \right)$$

Cf the Cauchy-Riemann equations! Introduce the 'complex' field ψ.

$$\psi = a e_1 = u + I v$$

Analytic if satisfies

$$\nabla \psi = 0$$

The key to analytic function theory. Generalises:

- 3-d, ψ even-grade multivector. Get spin harmonics — Pauli and Dirac electron wavefunctions.
- Spacetime, have

$$\nabla = e^0 \partial_t + e^i \partial_{x^i}, \qquad i = 1 \ldots 3.$$

ψ an even-grade multivector, $\nabla\psi = 0$ is neutrino wave equation. Add a mass term, get Dirac equation.

- Restrict ψ to pure bivector F, $\nabla F = 0$ is *all* free-field Maxwell equations

All examples of same mathematics

ANALYTIC FUNCTIONS

$$z = e_1 x = \mathrm{x} + I\mathrm{y}$$

$$z^* = \mathrm{x} - I\mathrm{y} = x e_1 = e_1(-e_2 x e_2)$$

Complex derivatives have properties

$$\partial_z z = 1 \qquad \partial_z z^* = 0$$

$$\partial_{z^*} z = 0 \qquad \partial_{z^*} z^* = 1$$

From these we have

$$\partial_z = \tfrac{1}{2}(\partial_\mathrm{x} - I\partial_\mathrm{y}), \quad \partial_{z^*} = \tfrac{1}{2}(\partial_\mathrm{x} + I\partial_\mathrm{y})$$

An analytic function depends on z, $\psi(\mathrm{x} + I\mathrm{y}) = \psi(z)$ — independent of z^*, so

$$\partial_{z^*}\psi(z) = 0$$

This is what the limit argument is all about! Equivalent to

$$\tfrac{1}{2}\left(\frac{\partial}{\partial \mathrm{x}} + I\frac{\partial}{\partial \mathrm{y}}\right)\psi = \tfrac{1}{2} e_1 \nabla\psi = 0$$

Recovering key equation. Solutions to $\nabla\psi = 0$ constructed as a power series in z:

$$\nabla z = \nabla(e_1 x) = 2e_1 \cdot \nabla x - e_1 \nabla x$$
$$= 2e_1 - 2e_1 = 0$$

Drives most of analytic function theory! *Eg*.

$$\nabla(e_1 x - z_0)^n = n\nabla(e_1 x - z_0)(e_1 x - z_0)^{n-1} = 0$$

Taylor expansion in z about z_0 is analytic

Problems

1. ∇ mapped to same algebra as ψ by e_1. Only works in 2-d $\Rightarrow$ Keep ∇ and ψ distinct.

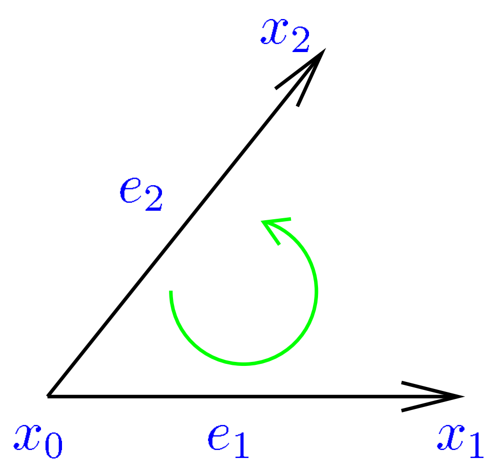
2. 'Limit' argument does not generalise. $\Rightarrow$ Replace with $\nabla\psi = 0$.

DIRECTED INTEGRATION THEORY

Familiar with divergence theorem, Green's theorem, Stokes' theorem and Cauchy integral formula. All special cases of a single integral formula!

Work in 2-d (generalises easily). Multivector-valued function $M(x)$ at points x_0, x_1, x_2.

$$e_1 = x_1 - x_0, \qquad e_2 = x_2 - x_0$$



Vector derivative at x_0 is

$$\nabla M = \lim_{x_i \mapsto x_0} e^1(M_1 - M_0) + e^2(M_2 - M_0)$$

where $M_i = M(x_i)$. Want to relate to a surface integral.

Extrapolate M linearly between base points

$$m(x) = M_0 + \sum_{i=1}^{2} (x - x_0)\cdot e^i \,(M_i - M_0)$$

Calculate integral of $m(x)$ around the perimeter (exercise)

$$\oint dS\, m(x) = e_1(M_0 - M_2) + e_2(M_1 - M_0)$$

$$= e_2 \wedge e_1 [e^1(M_1 - M_0) + e^2(M_2 - M_0)]$$

Now have

$$\nabla M = \lim_{x_i \mapsto x_0} \oint dS\, e^2 \wedge e^1\, m$$

But $e^2 \wedge e^1 = (IV)^{-1}$, where V is scalar area of triangle.

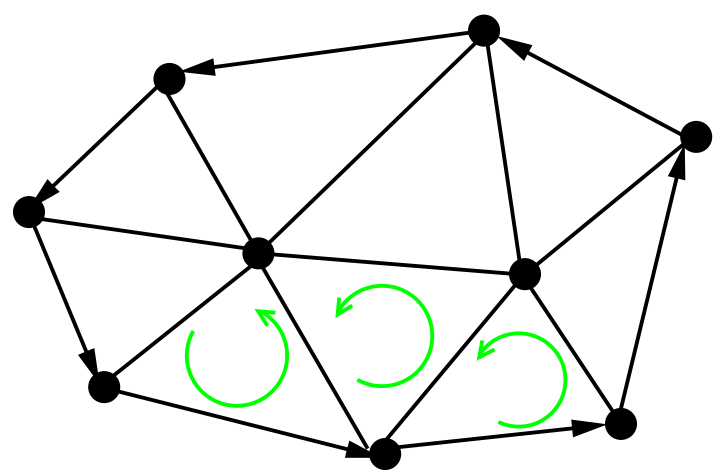
Take limit, replace m by M,

$$\nabla M = \lim_{V \mapsto 0} \frac{1}{V} \oint dS\, I^{-1} M$$

dS = RH surface element, I = RH pseudoscalar. dSI^{-1} vector-valued. Holds in any dimension. An alternative definition of ∇ — the derivative from an integral!

The Fundamental Theorem of Calculus

Build up over triangulated surface.



Interior lines cancel, left with

$$\int_V \dot{\nabla}\, dX\, \dot{M} = \oint_{\partial V} dS\, M$$

$dX = IdV$ is directed volume element. dS is directed surface element. Order is important, as is handedness.

This is the fundamental theorem of calculus. Relates the integral of a derivative to surface integral.

Reversed result:

$$\int_V \dot{M}\dot{\nabla}\, dX = \oint_{\partial V} M\, dS$$

Most general from

$$\int_V \dot{\mathsf{L}}(\dot{\nabla}\, dX) = \oint_{\partial V} \mathsf{L}(dS)$$

$\mathsf{L}(A)$ and multilinear function, grade $n-1$ multivector as argument. Holds in any dimension. Also holds for curved surfaces!

As example, recover divergence theorem. Let

$$\mathsf{L}(A) = \langle I^{-1} J A \rangle$$

where J is a vector. Find that

$$\int_V \langle \dot{J}\dot{\nabla} dX I^{-1} \rangle = \int_V \nabla\cdot J\, dV = \oint_{\partial V} \langle dS I^{-1} J \rangle$$

Normal to surface defined by

$$n\, dA = dS\, I^{-1}$$

dA a scalar measure. Points outwards in Euclidean spaces. More complicated with mixed signatures. Recover

$$\int_V \nabla\cdot J\, dV = \oint_{\partial V} n\cdot J\, dA$$

as expected. We can similarly go on to recover Green's theorem.

CAUCHY'S THEOREM RECOVERED

Return to 2-d ψ 'complex' valued. Fundamental theorem:

$$\int \nabla\psi \, dX = \oint dS \, \psi$$

But $z = e_1 x$ so write as

$$\oint \psi \, dz = \int e_1 \nabla \psi \, dX$$

Suppose take

$$\psi = \frac{f(z)}{(z - z_0)}, \qquad \nabla f = 0$$

Need properties of $(z - z_0)^{-1}$ (The Cauchy kernel). Have

$$\frac{1}{z - z_0} = \frac{(z - z_0)^*}{|(z - z_0)|^2} = \frac{x - x_0}{(x - x_0)^2} e_1$$

where $x_0 = e_1 z_0$. But 2-d Green's function $\ln|x - x_0|$ has

$$\nabla \ln|x - x_0| = \frac{x - x_0}{(x - x_0)^2}$$

Hence the Cauchy kernel satisfies

$$\nabla \frac{1}{z - z_0} = \nabla^2 \ln|x - x_0| e_1 = 2\pi\delta(x - x_0) e_1$$

$(z - z_0)^{-1}$ is Green's function for the vector derivative!

Put together:

$$\oint \frac{f(z)}{z-z_0} dz = e_1 \int \nabla \left(\frac{x-x_0}{(x-x_0)^2} e_1 f(x) \right) dX$$

$$= e_1 \int 2\pi\delta(x-x_0) e_1 f(x) I \, |dx| = 2\pi I f(z_0)$$

Recovers the Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi I} \oint \frac{f(z)}{z-z_0} dz$$

Now understand what each term is doing!

- dz is a tangent vector. Forms a geometric product in the integral.
- $(z-z_0)^{-1}$ is Green's function for the vector derivative ∇. Generates a δ-function at z_0.
- I (or i) is pseudoscalar from directed volume element $dX = I \, dV$.

Also understand residue term in Laurent expansion

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} \cdots \frac{a_{-1}}{z-z_0} + \sum_{i=0} a_i (z-z_0)^i$$

This is simply a weighted Greens function. Residue theorem recovers weight. Unites poles and residues with Green's functions and δ-functions.

Arbitrary Dimensions

Extend to arbitrary (Euclidean) dimensions. M is an even-grade multivector satisfying $\nabla M = 0$. The Green's function for ∇ is

$$G(x, x_0) = \frac{1}{S_n} \frac{x - x_0}{|x - x_0|^n}$$

S_n = surface area of $(m-1)$-dimensional unit ball. The Green's function satisfies

$$\nabla G(x, x_0) = \dot{G}(x, x_0)\dot{\nabla} = \delta(x - x_0)$$

A version of Cauchy's theorem in n-d constructed from

$$\oint_{\partial V} \frac{x - x_0}{|x - x_0|^n} dS\, M = \int_V \left(\frac{x - x_0}{|x - x_0|^n} \right) \overleftarrow{\nabla}\, dX\, M$$
$$+ \int_V \frac{x - x_0}{|x - x_0|^n} \dot{\nabla}\, dX\, \dot{M}$$

where use $\overleftarrow{\nabla}$ for ∇ acting on object to left. Since M commutes with dX, final term vanishes, leaving

$$M(x_0) = \frac{1}{IS_n} \oint_{\partial V} \frac{x - x_0}{|x - x_0|^n} dS\, M$$