February 9, 1999

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 8

SUMMARY

In this lecture we introduce the spacetime algebra, the geometric algebra of spacetime. This forms the basis for most of the remaining course, and is central to the formulation of a powerful gauge theory of gravity.

- Adding a vector for time – the 4-d spacetime algebra and some consequences of a mixed signature metric.
- Paths, observers and frames.
- Projective splits for observers.
- Handling Lorentz transformations with rotors.
- Photons and redshifts.
- The structure of the Lorentz group.

An Algebra for Spacetime

Aim — to construct the geometric algebra of spacetime.

Invariant interval is

$$s^2 = c^2t^2 - x^2 - y^2 - z^2$$

(The 'particle physics' choice. GR flips signs. No observable consequences). Work in natural units, $c = 1$.

Need four vectors $\{e_0, e_i\}, i = 1 \ldots 3$ with properties

$$e_0{}^2 = 1, \qquad e_i{}^2 = -1$$

$$e_0 \cdot e_i = 0, \qquad e_i \cdot e_j = -\delta_{ij}$$

Summarised by

$$e_\mu \cdot e_\nu = \text{diag}(+ - - -)\,, \quad \mu, \nu = 0 \ldots 3$$

Bivectors

$4 \cdot 3/2 = 6$ bivectors in algebra. Two types

1. Those containing e_0, *e.g.* $\{e_i \wedge e_0\}$,
2. Those not containing e_0, *e.g.* $\{e_i \wedge e_j\}$.

For any pair of vectors a and b, $a \cdot b = 0$ have

$$(a \wedge b)^2 = abab = -abba = -a^2b^2$$

The two types have different squares

$$(e_i \wedge e_j)^2 = -e_i{}^2 e_j{}^2 = -1$$

Spacelike Euclidean bivectors, generate rotations in a plane.

$$(e_i \wedge e_0)^2 = -e_i{}^2 e_0{}^2 = 1$$

Timelike bivectors. Generate hyperbolic geometry:

$$\begin{aligned} e^{\alpha e_1 e_0} &= 1 + \alpha e_1 e_0 + \alpha^2/2! + \alpha^3/3!\, e_1 e_0 + \cdots \\ &= \mathrm{ch}(\alpha) + \mathrm{sh}(\alpha) e_1 e_0 \end{aligned}$$

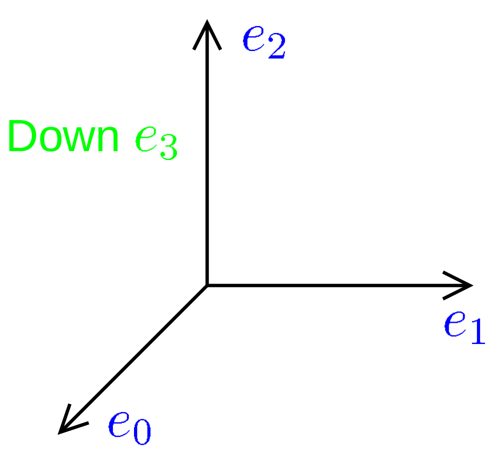
Crucial to treatment of Lorentz transformations.

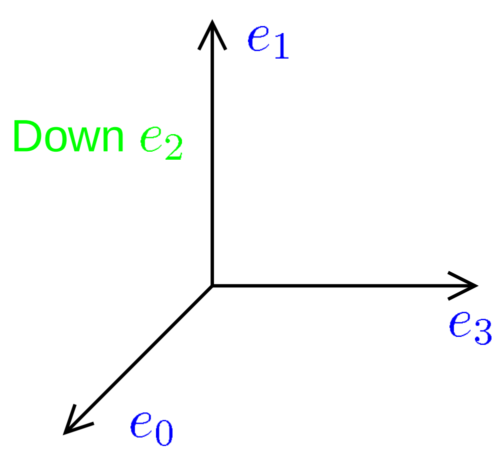
THE PSEUDOSCALAR

Define the pseudoscalar I

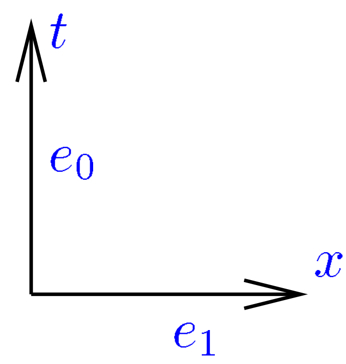
$$I = e_0 e_1 e_2 e_3$$

Still chosen to be right-handed. Projecting onto subspaces have





Have to be careful with these definitions. Traditionally draw spacetime diagrams as



'right-handed' volume element for this is $e_1 e_0$.

Since I is grade 4, it has

$$\tilde{I} = e_3 e_2 e_1 e_0 = I$$

Compute the square of I :

$$I^2 = I\tilde{I} = (e_0 e_1 e_2 e_3)(e_3 e_2 e_1 e_0) = -1$$

Multiply bivector by I, get grade $4 - 2 = 2$ — another bivector. Provides map between bivectors with positive and negative square:

$$I e_1 e_0 = e_1 e_0 I = e_1 e_0 e_0 e_1 e_2 e_3 = -e_2 e_3$$

Define $B_i = e_i e_0$. Bivector algebra is

$$B_i \times B_j = \epsilon_{ijk} \, I B_k, \quad (IB_i) \times (IB_j) = -\epsilon_{ijk} \, I B_k$$
$$(IB_i) \times B_j = -\epsilon_{ijk} B_k$$

Have four vectors, and four trivectors in algebra. Interchanged by duality

$$e_1e_2e_3 = e_0e_0e_1e_2e_3 = e_0I = -Ie_0$$

NB I anticommutes with vectors and trivectors. (In space of even dimensions). I always commutes with even-grade.

The Spacetime algebra

Putting terms together, get an algebra with 16 terms:

1	$\{\gamma_\mu\}$	$\{\gamma_\mu \wedge \gamma_\nu\}$	$\{I\gamma_\mu\}$	I
1 scalar	4 vectors	6 bivectors	4 trivectors	1 pseudoscalar

The spacetime algebra or STA. Use $\{\gamma_\mu\}$ for preferred orthonormal frame. Also define

$$\boldsymbol{\sigma}_i = \gamma_i\gamma_0$$

Not used i for the pseudoscalar. Potentially confusing. The $\{\gamma_\mu\}$ satisfy

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}$$

This is the Dirac matrix algebra (with identity matrix on right). A matrix representation of the STA. Explains notation, but $\{\gamma_\mu\}$ are vectors, not a set of matrices in 'isospace'.

Frames and Trajectories

$x(\lambda)$ a spacetime trajectory. Tangent vector is

$$x' = \frac{x(\lambda)}{\partial \lambda}$$

Two cases to consider:

Timelike, $x'^2 > 0$. Introduce proper time τ:

$$v = \partial_\tau x = \dot{x}, \qquad v^2 = 1$$

Observers measure this. Unit vector v defines the instantaneous rest frame.

Null, $x'^2 = 0$. Describes a null trajectory. Taken by massless particles, (photons, *etc.*). Proper distance/time = 0. Photons do carry an intrinsic clock (their frequency), but can tick at arbitrary rate.

Now take observer on timelike path with instantaneous velocity v. What do we measure? Construct a rest frame $\{e_i\}$,

$$e_i \cdot v = 0, \qquad i = 1 \ldots 3$$

Take point on worldline as spatial origin. Event x has time coordinate $t = x \cdot v$ and space coordinates

$$x_i = x \cdot e_i$$

The 3-d vector to a point on the worldline of an object intersecting our rest frame:

$$x_i e^i = x\cdot e_\mu \, e^\mu - x\cdot e_0 \, e^0 = x - x\cdot v \, v = x\wedge v \, v$$

Wedge product with v projects onto components of x in rest frame of v. Define relative vector by spacetime bivector $x\wedge v$:

$$\boldsymbol{x} = x\wedge v$$

With this definitions have

$$xv = x\cdot v + x\wedge v = t + \boldsymbol{x}$$

Invariant distance decomposes as

$$\begin{aligned} x^2 = xvvx &= (x\cdot v + x\wedge v)(x\cdot v + v\wedge x) \\ &= (t + \boldsymbol{x})(t - \boldsymbol{x}) = t^2 - \boldsymbol{x}^2 \end{aligned}$$

Recovers usual result. Built into definition of STA.

The Even Subalgebra

Each observer sees set of relative vectors. Model these as spacetime bivectors. Take timelike vector γ_0, relative vectors $\boldsymbol{\sigma}_i = \gamma_i \gamma_0$. Satisfy

$$\begin{aligned} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j &= \tfrac{1}{2}(\gamma_i\gamma_0\gamma_j\gamma_0 + \gamma_j\gamma_0\gamma_i\gamma_0) \\ &= \tfrac{1}{2}(-\gamma_i\gamma_j - \gamma_j\gamma_i) = \delta_{ij} \end{aligned}$$

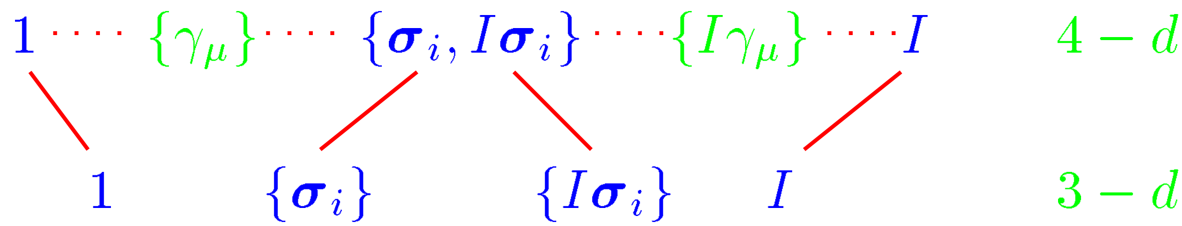
Generators for a 3-d algebra! The GA of the 3-d relative space in rest frame of γ_0. Volume element

$$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = (\gamma_1\gamma_0)(\gamma_2\gamma_0)(\gamma_3\gamma_0) = -\gamma_1\gamma_0\gamma_2\gamma_3 = I$$

so 3-d subalgebra shares same pseudoscalar as spacetime. Still have

$$\tfrac{1}{2}(\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j\boldsymbol{\sigma}_i) = \epsilon_{ijk} I \boldsymbol{\sigma}_k$$

relative vectors and relative bivectors are spacetime bivectors. Projected onto the even subalgebra of the STA.



The 6 spacetime bivectors split into relative vectors and relative bivectors. This split is observer dependent. A very useful technique.

Conventions

Expression like $\boldsymbol{a}\wedge\boldsymbol{b}$ potentially confusing.

- Spacetime bivectors used as relative vectors are written in bold. Includes the $\{\boldsymbol{\sigma}_i\}$.
- If both arguments bold, dot and wedge symbols drop down to their 3-d meaning.
- Otherwise, keep spacetime definition.

EXAMPLES

i. Velocity

Observer, with constant velocity v. Measures relative velocity of a particle with proper velocity $u(\tau)$, $u^2 = 1$. Form

$$uv = \partial_\tau (x(\tau)v) = \partial_\tau (t + \boldsymbol{x})$$

So that

$$\partial_\tau t = u \cdot v$$

The relative velocity is

$$\boldsymbol{u} = \frac{\partial \boldsymbol{x}}{\partial t} = \frac{\partial \boldsymbol{x}}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{u \wedge v}{u \cdot v}$$

Familiar — same as used in projective geometry! Also ensured that projective vectors have positive square. Use this computer vision applications!

ii. Momentum and Wave Vectors

Observe particle with energy-momentum p. Energy measured $= p \cdot v$, relative momentum $= p \wedge v$,

$$pv = p \cdot v + p \wedge v = E + \boldsymbol{p}$$

Recover the invariant

$$m^2 = p^2 = pvvp = (E + \boldsymbol{p})(E - \boldsymbol{p}) = E^2 - \boldsymbol{p}^2$$

Similarly, for a photon wave-vector k,

$$kv = k\cdot v + k\wedge v = \omega + \boldsymbol{k}$$

For photons in empty space $k^2 = 0$ so

$$0 = kvvk = (\omega + \boldsymbol{k})(\omega - \boldsymbol{k}) = \omega^2 - \boldsymbol{k}^2$$

Recovers $|\boldsymbol{k}| = \omega$. Holds in all frames.

LORENTZ TRANSFORMATIONS

Usually expressed as a coordinate transformation, *e.g.*

$$\mathrm{x}' = \gamma(\mathrm{x} - \beta t) \qquad t' = \gamma(t - \beta \mathrm{x})$$
$$\mathrm{x} = \gamma(\mathrm{x}' + \beta t') \qquad t = \gamma(t' + \beta \mathrm{x}')$$

where $\gamma = (1-\beta^2)^{-1/2}$ and β is scalar velocity. Vector x decomposed in two frames, $\{e_\mu\}$ and $\{e'_\mu\}$,

$$x = x^\mu e_\mu = x^{\mu\prime} e'_\mu$$

with

$$t = e^0\cdot x, \qquad t' = e^{0\prime}\cdot x.$$

Concentrating on the 0, 1 components:

$$te_0 + \mathrm{x}e_1 = t'e'_0 + \mathrm{x}'e'_1,$$

Derive vector relations

$$e'_0 = \gamma(e_0 + \beta e_1), \quad e'_1 = \gamma(e_1 + \beta e_0).$$

Gives new frame in terms of the old. Now introduce 'hyperbolic angle' α,

$$\tanh\alpha = \beta, \qquad (\beta < 1),$$

Gives

$$\gamma = (1 - \tanh^2\alpha)^{-1/2} = \cosh\alpha.$$

Vector e_0' is now

$$\begin{aligned} e_0' &= \mathrm{ch}(\alpha)e_0 + \mathrm{sh}(\alpha)e_1 \\ &= (\mathrm{ch}(\alpha) + \mathrm{sh}(\alpha)e_1e_0)e_0 = \mathrm{e}^{\alpha\, e_1 e_0}\, e_0, \end{aligned}$$

Similarly, we have

$$e_1' = \mathrm{ch}(\alpha)e_1 + \mathrm{sh}(\alpha)e_0 = \mathrm{e}^{\alpha\, e_1 e_0}\, e_1.$$

Two other frame vectors unchanged. Relationship between the frames is

$$e_\mu' = Re_\mu\tilde{R}, \quad e^{\mu\prime} = Re^\mu\tilde{R}, \quad R = \mathrm{e}^{\alpha\, e_1 e_0/2}.$$

Same rotor prescription works for boosts as well as rotations! Spacetime is a unified entity now.

EXAMPLES

i. Addition of Velocities

Two objects separating, velocities

$$v_1 = \mathrm{e}^{\alpha_1 e_1 e_0}\, e_0, \qquad v_2 = \mathrm{e}^{-\alpha_2 e_1 e_0}\, e_0.$$

What is the relative velocity sees for each other? Form

$$\frac{v_1 \wedge v_2}{v_1 \cdot v_2} = \frac{\langle \, e^{(\alpha_1 + \alpha_2) e_1 e_0} \, \rangle_2}{\langle \, e^{(\alpha_1 + \alpha_2) e_1 e_0} \, \rangle_0} = \frac{\sinh(\alpha_1 + \alpha_2) e_1 e_0}{\cosh(\alpha_1 + \alpha_2)}.$$

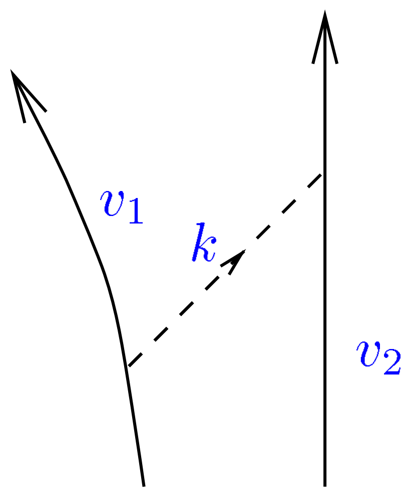
Both observers measure relative velocity

$$\tanh(\alpha_1 + \alpha_2) = \frac{\tanh\alpha_1 + \tanh\alpha_2}{1 - \tanh\alpha_1 \tanh\alpha_2}$$

Addition of velocities is achieved by adding hyperbolic angles. Recovers familiar formula.

ii. Photons and Redshifts

Two particles on different worldlines. Particle 1 emits a photon, received by particle 2



Frequency for particle 1 is $\omega_1 = v_1 \cdot k$, for particle 2 is $\omega_2 = v_2 \cdot k$. Ratio describes the Doppler effect, often expressed as a redshift:

$$1 + z = \omega_1 / \omega_2$$

Can be applied in many ways. If emitter receding in e_1 direction, and $v_2 = e_0$, have

$$k = \omega_2(e_0 + e_1), \qquad v_1 = \cosh\alpha\, e_0 - \sinh\alpha\, e_1$$

so that

$$1 + z = \frac{\omega_2(\cosh\alpha + \sinh\alpha)}{\omega_2} = \mathrm{e}^{\alpha}$$

Boost of a null vector = dilation. Just as in $\mathcal{G}_{n,n}$! Velocity of emitter in e_0 frame is $\tanh\alpha$, and

$$\mathrm{e}^{\alpha} = \left(\frac{1 + \tanh\alpha}{1 - \tanh\alpha}\right)^{1/2}$$

Aberration formulae obtained same way.

The Lorentz Group

Group of transformations preserving lengths and angles. Build from reflections

$$a \mapsto -nan^{-1}$$

The n^{-1} needed for both timelike $n^2 > 0$ and spacelike $n^2 < 0$. Cannot have null n. Timelike n generates time-reversal. Spacelike n preserve time ordering. Full Lorentz group contains 4 sectors.

		Space Reflection
	I Proper Orthochronous	II I with space refelection
Time reversal	III I with time reversal	IV I with $a \mapsto -a$

Easily understood in STA. Combine an even numbers of reflections,

$$a \mapsto \psi a \psi^{-1}$$

ψ is an even multivector. Need to ensure that result is a vector. Form

$$\psi\tilde{\psi} = (\psi\tilde{\psi})^{\sim}$$

Even and equal to own reverse. A scalar and a pseudoscalar

$$\psi\tilde{\psi} = \alpha_1 + I\alpha_2 = \rho\, \mathrm{e}^{I\beta}$$

with $\rho \neq 0$. Define rotor R by

$$R = \psi(\rho\, \mathrm{e}^{I\beta})^{-1/2}$$

so that

$$R\tilde{R} = \psi\tilde{\psi}(\rho\, \mathrm{e}^{I\beta})^{-1} = 1$$

as required. Now have

$$\psi = \rho^{1/2}\, \mathrm{e}^{I\beta/2}\, R, \qquad \psi^{-1} = \rho^{-1/2}\, \mathrm{e}^{-I\beta/2}\, \tilde{R}$$

and transformation becomes

$$a \mapsto \mathrm{e}^{I\beta/2} Ra\, \mathrm{e}^{-I\beta/2} \tilde{R} = \mathrm{e}^{I\beta} Ra\tilde{R}$$

Must have $\beta = 0$ or $\beta = \pi$. Gives

$$a \mapsto \pm Ra\tilde{R}$$

Proper Orthochronous Transformations

Transformation $a \mapsto Ra\tilde{R}$ preserves causal ordering. Take $\gamma_0 \mapsto v = R\gamma_0\tilde{R}$. Need the γ_0 component of v to be positive,

$$\gamma_0 \cdot v = \langle \gamma_0 R \gamma_0 \tilde{R} \rangle > 0$$

Decomposing in γ_0 frame

$$R = \alpha + \boldsymbol{a} + I\boldsymbol{b} + I\beta$$

find that

$$\langle \gamma_0 R \gamma_0 \tilde{R} \rangle = \alpha^2 + \boldsymbol{a}^2 + \boldsymbol{b}^2 + \beta^2 > 0$$

as required. Rotor transformation law define the restricted Lorentz group. Physically most relevant. $\beta = \pi$ gives class IV transformations.