February 11, 1999

# PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 9

## SUMMARY

In this lecture we will concentrate on the rotor representation of Lorentz transformations. An analogy with rigid-body mechanics leads to a new rotor-based technique for analysing the relativistic equations of motion of a point particle.

- Fixed points and the celestial sphere.
- Pure boosts and acceleration as a bivector.
- Relativistic equations of motion for a point particle described by rotors.
- Thomas precession.
- The Lorentz force law and the Faraday bivector.
- Point particle in a constant field.
- A classical model of $g = 2$.

# Invariant Decomposition

Restricted Lorentz transformation $a \mapsto Ra\tilde{R}$. Every spacetime rotor can be written as

$$R = \pm\, \mathrm{e}^{B/2}$$

The minus sign rarely needed. Can find Lorentz invariant decomposition. Write

$$B^2 = \langle B^2\rangle_0 + \langle B^2\rangle_4 = \rho\, \mathrm{e}^{I\phi}$$

(assume $\rho \neq 0$). Define

$$\hat{B} = \rho^{-1/2}\, \mathrm{e}^{-I\phi/2}\, B$$

So that

$$\hat{B}^2 = \rho^{-1}\, \mathrm{e}^{-I\phi}\, B^2 = 1$$

Now have

$$B = \rho^{1/2}\, \mathrm{e}^{I\phi/2}\, \hat{B} = \alpha\hat{B} + \beta I\hat{B}$$

Since

$$\hat{B}\, I\hat{B} = I\hat{B}\, \hat{B} = I$$

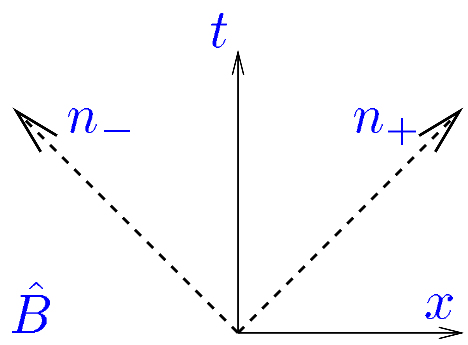
Have commuting blades $\alpha\hat{B}$, $\beta I\hat{B}$. Write

$$R = \mathrm{e}^{\alpha\hat{B}/2}\, \mathrm{e}^{\beta I\hat{B}/2} = \mathrm{e}^{\beta I\hat{B}/2}\, \mathrm{e}^{\alpha\hat{B}/2}$$

*Invariant* split into a boost and a rotation.

# FIXED POINTS

Timelike bivector $\hat{B}$, $\hat{B}^2 = 1$, has two null vectors $n_\pm$.




Satisfy (exercise)

$$\hat{B}\cdot n_\pm = \pm n_\pm$$

Necessarily null, since

$$(\hat{B}\cdot n_\pm)\cdot n_\pm = 0 = \pm n_\pm^2$$

$n_\pm$ chosen so that

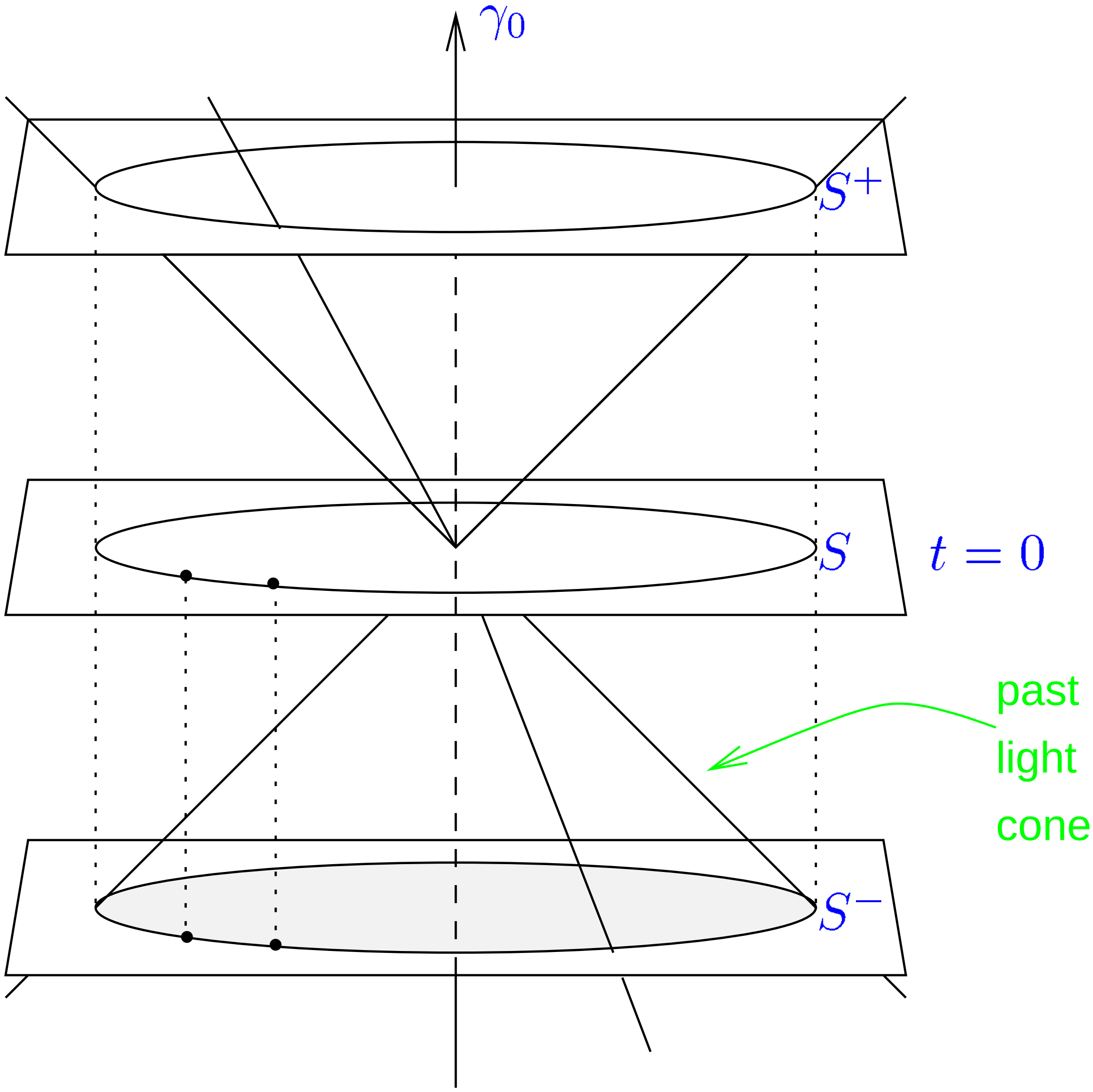
$$n_+ \wedge n_- = 2\hat{B}$$

Form a null basis for $\hat{B}$ plane. $n_\pm$ anticommute with $\hat{B}$, commute with $I\hat{B}$. Lorentz transformation gives:

$$Rn_\pm\tilde{R} = \mathrm{e}^{\alpha\hat{B}/2}\, n_\pm\, \mathrm{e}^{-\alpha\hat{B}/2} = \mathrm{e}^{\alpha\hat{B}}\, n_\pm$$
$$= \mathrm{ch}(\alpha)n_\pm + \mathrm{sh}(\alpha)\hat{B}\cdot n_\pm = \mathrm{e}^{\pm\alpha}\, n_\pm$$

Null directions are scaled. Rotors for dilations again.

# The Celestial Sphere




Visualise Lorentz transformations through effect on the past light sphere — the celestial sphere $S^-$.

Observer $\gamma_0$ receives light along null vector $n$. Form relative vector $n\wedge\gamma_0$, with

$$(n\wedge\gamma_0)^2 = (n\cdot\gamma_0)^2 - n^2\gamma_0^2 = (n\cdot\gamma_0)^2$$

Form projective unit vector $\boldsymbol{n}$

$$\boldsymbol{n} = n\wedge\gamma_0/n\cdot\gamma_0$$

Maps all past events onto a sphere. Second observer $v$ forms vectors $n\wedge v/n\cdot v$. Transform back to $\gamma_0$ frame for comparison:

$$\boldsymbol{n}' = \tilde{R}\frac{n\wedge v}{n\cdot v}R = \frac{n'\wedge\gamma_0}{n'\cdot\gamma_0}$$

$n' = \tilde{R}nR$. See effect by moving points on sphere with $n \mapsto \tilde{R}nR$. Two fixed points.

## Pure Boosts and Observer Splits

Velocity $u$ boosted $v$. Need rotor with no additional rotation component:

$$v = Lu\tilde{L}$$

$La_\perp\tilde{L} = a_\perp$ for $a_\perp$ outside $u\wedge v$. Bivector generator is multiple of $u\wedge v$. Anticommutes with $u$ and $v$, so

$$v = Lu\tilde{L} = L^2u \qquad \Rightarrow L^2 = vu$$

Solution is (check!)

$$L = \frac{1+vu}{[2(1+u\cdot v)]^{1/2}} = \exp\left(\frac{\alpha}{2}\frac{v\wedge u}{|v\wedge u|}\right)$$

where $\mathrm{ch}(\alpha) = u\cdot v$.

Now take arbitrary rotor $R$. Decompose in $\gamma_0$ frame, $v = R\gamma_0\tilde{R}$. Pure boost is

$$L = \frac{1 + v\gamma_0}{[2(1 + v\cdot\gamma_0)]^{1/2}} = \exp\left(\frac{\alpha}{2}\frac{v\wedge\gamma_0}{|v\wedge\gamma_0|}\right)$$

$v\cdot\gamma_0 = \mathrm{ch}(\alpha)$. Define rotor $U$,

$$U = \tilde{L}R, \qquad U\tilde{U} = 1$$

Satisfies

$$U\gamma_0\tilde{U} = \tilde{L}vL = \gamma_0$$

so $U\gamma_0 = \gamma_0 U$, and $U = e^{I\boldsymbol{b}/2}$ — a pure rotation in $\gamma_0$ frame.

$$R = LU$$

Frame dependent decomposition. Do not commute.

## Spacetime Rotor Equations

Trajectory $x(\tau)$, future-pointing velocity $v = \partial_\tau x, v^2 = 1$. Cf rigid-body dynamics. Write

$$v = R\gamma_0\tilde{R}$$

Put dynamics in rotor $R$! Compute acceleration

$$\dot{v} = \partial_\tau(R\gamma_0\tilde{R}) = \dot{R}\gamma_0\tilde{R} + R\gamma_0\dot{\tilde{R}}$$

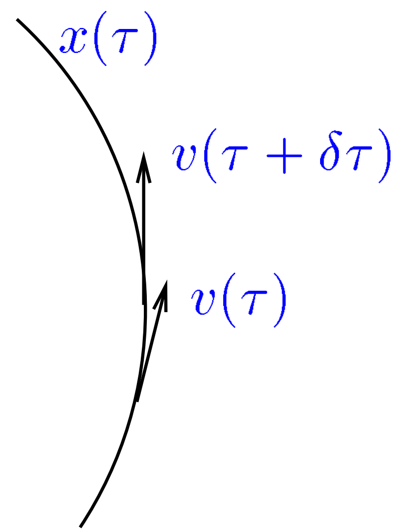
$\dot{R}\tilde{R}$ is a bivector. Have have

$$\dot{v} = \dot{R}\tilde{R}v - v\dot{R}\tilde{R} = 2(\dot{R}\tilde{R})\cdot v$$

Consistent with $v\cdot\dot{v} = 0$. Now have

$$\dot{v}v = 2(\dot{R}\tilde{R})\cdot vv$$

But rotor $R$ can carry extra rotation. Want rotor to be pure boost at each instant




To first order

$$v(\tau + \delta\tau) = v(\tau) + \delta\tau\,\dot{v}$$

Proper rotor between $v(\tau)$ and $v(\tau + \delta\tau)$ is

$$L = \frac{1 + v(\tau + \delta\tau)v(\tau)}{[2(1 + v(\tau + \delta\tau)\cdot v(\tau))]^{1/2}} = 1 + \tfrac{1}{2}\delta\tau\,\dot{v}v$$

But since

$$v(\tau + \delta\tau) = R(\tau + \delta\tau)\gamma_0\tilde{R}(\tau + \delta\tau) = LR(\tau)\gamma_0\tilde{R}(\tau)\tilde{L}$$

must set

$$R(\tau + \delta\tau) = R(\tau) + \delta\tau\,\dot{R} = LR(\tau)$$

Correct expression is

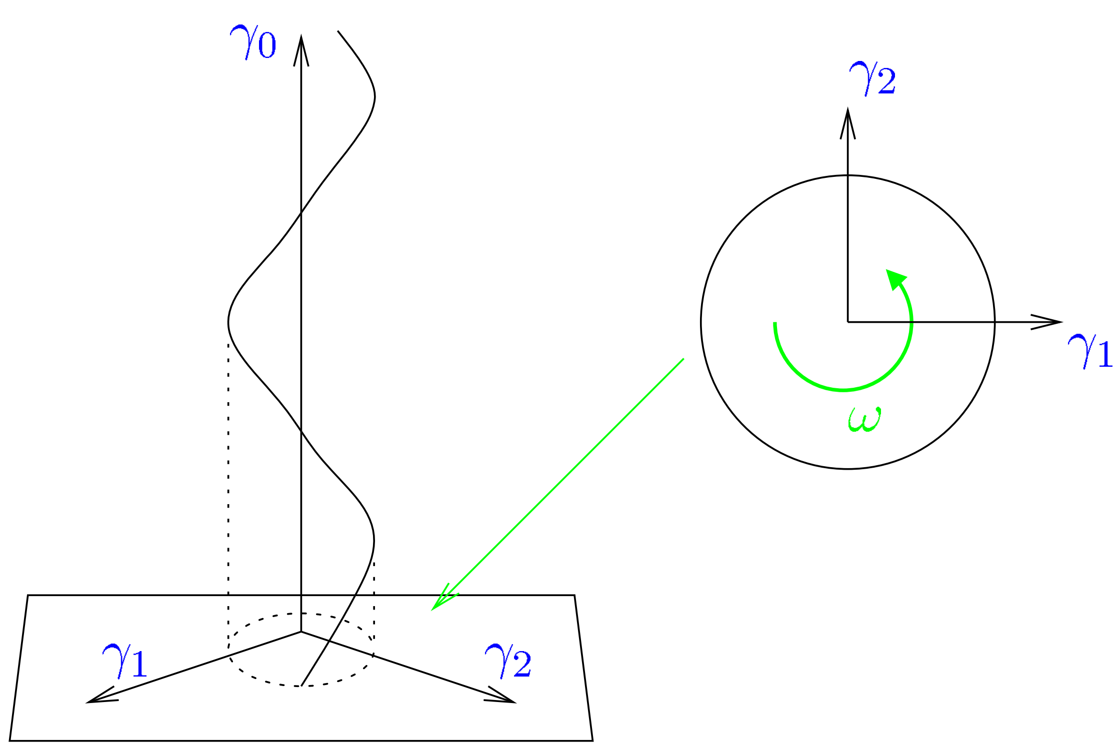
$$\dot{R}\tilde{R} = \tfrac{1}{2}\dot{v}v$$

$\dot{v}v$ is acceleration bivector — $\dot{v}$ in instantaneous rest frame. This is generator for $R$.

## Thomas Precession

Particle on circular orbit. Worldline

$$x(\tau) = t(\tau)\gamma_0 + a[\cos(\omega t)\gamma_1 + \sin(\omega t)\gamma_2]$$



Velocity is

$$v = \partial_\tau x = \dot{t}\,(\gamma_0 + aw[-\sin(\omega t)\gamma_1 + \cos(\omega t)\gamma_2])$$

Relative velocity $\boldsymbol{v} = v\wedge\gamma_0/v\cdot\gamma_0$ has $|\boldsymbol{v}| = a\omega$. Define

$$\tanh\alpha = a\omega, \qquad \dot{t} = \cosh\alpha.$$

Velocity now

$$\begin{aligned} v &= \mathrm{ch}(\alpha)\gamma_0 + \mathrm{sh}(\alpha)[-\sin(\omega t)\gamma_1 + \cos(\omega t)\gamma_2] \\ &= \mathrm{e}^{\alpha\boldsymbol{n}/2}\,\gamma_0\,\mathrm{e}^{-\alpha\boldsymbol{n}/2} \end{aligned}$$

where

$$\boldsymbol{n} = -\sin(\omega t)\boldsymbol{\sigma}_1 + \cos(\omega t)\boldsymbol{\sigma}_2$$

Simplify with

$$\boldsymbol{n} = \mathrm{e}^{-\omega t I\boldsymbol{\sigma}_3}\,\boldsymbol{\sigma}_2 = R_\omega\boldsymbol{\sigma}_2\tilde{R}_\omega, \quad R_\omega = \exp(-\omega t I\boldsymbol{\sigma}_3/2)$$

Gives

$$\mathrm{e}^{\alpha\boldsymbol{n}/2} = \exp(\alpha R_\omega\boldsymbol{\sigma}_2\tilde{R}_\omega/2) = R_\omega\,\mathrm{e}^{\alpha\boldsymbol{\sigma}_2/2}\,\tilde{R}_\omega$$

Define $R_\alpha = \exp(\alpha\boldsymbol{\sigma}_2/2)$. Now have

$$v = R_\omega R_\alpha\tilde{R}_\omega\gamma_0 R_\omega\tilde{R}_\alpha\tilde{R}_\omega = R_\omega R_\alpha\gamma_0\tilde{R}_\alpha\tilde{R}_\omega,$$

Rotor for motion must have form

$$R = R_\omega R_\alpha R_T, \qquad R_T = \exp(-\omega_T t I\boldsymbol{\sigma}_3/2)$$

Determine $\omega_T$ from $\dot{v}v$. Write

$$v = R_\omega v_\alpha \tilde{R}_\omega, \qquad v_\alpha = R_\alpha \gamma_0 \tilde{R}_\alpha$$

Get

$$\begin{aligned} \dot{v}v &= R_\omega [2(\tilde{R}_\omega \dot{R}_\omega)\cdot v_\alpha \, v_\alpha] \tilde{R}_\omega \\ &= \omega \, \mathrm{sh}(\alpha)\mathrm{ch}(\alpha) R_\omega [-\mathrm{ch}(\alpha)\boldsymbol{\sigma}_1 + \mathrm{sh}(\alpha) I\boldsymbol{\sigma}_3] \tilde{R}_\omega \end{aligned}$$

Also need $2\dot{R}\tilde{R}$, goes as

$$\begin{aligned} &2\dot{R}_\omega \tilde{R}_\omega + 2R_\omega R_\alpha \dot{R}_T \tilde{R}_T \tilde{R}_\alpha \tilde{R}_\omega \\ &= \mathrm{ch}(\alpha) R_\omega [-\omega I\boldsymbol{\sigma}_3 - \omega_T R_\alpha I\boldsymbol{\sigma}_3 \tilde{R}_\alpha] \tilde{R}_\omega \\ &= \mathrm{ch}(\alpha) R_\omega [-(\omega + \omega_T \mathrm{ch}(\alpha)) I\boldsymbol{\sigma}_3 + \omega_T \mathrm{sh}(\alpha) \boldsymbol{\sigma}_1] \tilde{R}_\omega \end{aligned}$$

So $\omega_T = -\mathrm{ch}(\alpha)\omega$. Full rotor is

$$R = \mathrm{e}^{-\omega t I\boldsymbol{\sigma}_3/2} \, \mathrm{e}^{\alpha\boldsymbol{\sigma}_2/2} \, \mathrm{e}^{\mathrm{ch}(\alpha)\omega t I\boldsymbol{\sigma}_3/2}$$

$\omega_T \neq \omega \Rightarrow$ Thomas precession. Vector $\gamma_1$ transported round circle,

$$e_1 = R\gamma_1 \tilde{R}$$

After time $t = 2\pi/\omega$, vector transformed to

$$e_1(2\pi/\omega) = \mathrm{e}^{\alpha\boldsymbol{\sigma}_2/2} \, \mathrm{e}^{2\pi \mathrm{ch}(\alpha) I\boldsymbol{\sigma}_3} \, \gamma_1 \, \mathrm{e}^{-\alpha\boldsymbol{\sigma}_2/2}$$

Precessed through angle $\theta = 2\pi(\cosh\alpha - 1)$. Effect is order $|\boldsymbol{v}|^2/c^2$.

# The Lorentz Force Law

Familiar with non-relativistic form

$$\frac{d\boldsymbol{p}}{dt} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

All relative vectors in $\gamma_0$ frame, $\boldsymbol{E} = E_i \boldsymbol{\sigma}_i$ *etc*. Want relativistic version of law. Have

$$\boldsymbol{p} = p \wedge \gamma_0, \quad \dot{t} = v \cdot \gamma_0$$

$\times$ through by $v \cdot \gamma_0$. Get $\dot{p} \wedge \gamma_0$ on left. On right

$$v \cdot \gamma_0 \, \boldsymbol{E} = v \cdot (\boldsymbol{E} \wedge \gamma_0) - (v \cdot \boldsymbol{E}) \wedge \gamma_0 = (\boldsymbol{E} \cdot v) \wedge \gamma_0$$

and

$$-v \cdot \gamma_0 \, \boldsymbol{v} \cdot (I\boldsymbol{B}) = -(v \wedge \gamma_0) \times (I\boldsymbol{B})$$
$$= [(I\boldsymbol{B}) \cdot v] \wedge \gamma_0 + [\gamma_0 \cdot (I\boldsymbol{B})] \wedge v = [(I\boldsymbol{B}) \cdot v] \wedge \gamma_0$$

Used Jacobi identity at intermediate step. Now have

$$\frac{d\boldsymbol{p}}{d\tau} = \dot{p} \wedge \gamma_0 = q[(\boldsymbol{E} + I\boldsymbol{B}) \cdot v] \wedge \gamma_0$$

Define Faraday bivector $F$

$$F = \boldsymbol{E} + I\boldsymbol{B}$$

The covariant electromagnetic field strength. More next lecture!

$$\dot{p}\wedge\gamma_0 = q(F\cdot v)\wedge\gamma_0$$

Must hold in all frames, so remove $\gamma_0$. With $p = mv$, get relativistic form of Lorentz force law,

$$m\dot{v} = qF\cdot v$$

Manifestly Lorentz covariant. Acceleration bivector is

$$\dot{v}v = \frac{q}{m}F\cdot v\,v = \frac{q}{m}(F\cdot v)\wedge v = \frac{q}{m}\boldsymbol{E}_v$$

where $\boldsymbol{E}_v$ is relative electric field in the $v$ frame.

## ROTOR FORM OF LORENTZ FORCE LAW

Use $v = R\gamma_0\tilde{R}$,

$$\dot{v} = 2\dot{R}\tilde{R}v = 2(\dot{R}\tilde{R})\cdot v = \frac{q}{m}F\cdot v$$

Equate projected terms

$$\dot{R} = \frac{q}{2m}FR$$

Not most general, but simplest.

## Example - Constant Field

Easy now! Integrate rotor equation

$$R = \exp(\frac{q}{2m}F\tau)$$

Now do invariant decomposition of $F$

$$F^2 = \langle F^2 \rangle_0 + \langle F^2 \rangle_4 = \rho\, \mathrm{e}^{I\beta}$$

so that

$$F = \rho^{1/2}\, \mathrm{e}^{I\beta/2}\, \hat{F} = \alpha \hat{F} + I\beta \hat{F}$$

where $\hat{F}^2 = 1$. (For null $F$ use different procedure). Have

$$R = \exp(\frac{q}{2m}\alpha \hat{F}\tau) \exp(\frac{q}{2m} I\beta \hat{F}\tau)$$

Now decompose initial velocity $v_0 = \gamma_0$

$$v_0 = \hat{F}^2 v_0 = \hat{F}\, \hat{F}\cdot v_0 + \hat{F}\, \hat{F}\wedge v_0 = v_{0||} + v_{0\perp}$$

$v_{0||} = \hat{F}\, \hat{F}\cdot v_0$ anticommutes with $\hat{F}$, $v_{0\perp}$ commutes, so

$$\dot{x} = \exp(\frac{q}{m}\alpha \hat{F}\tau) v_{0||} + \exp(\frac{q}{m} I\beta \hat{F}\tau) v_{0\perp}$$

Now integrate to get the particle history

$$x - x_0 = \frac{\mathrm{e}^{q\alpha \hat{F}\tau/m} - 1}{q\alpha/m} \hat{F}\cdot v_0 + \frac{\mathrm{e}^{q\beta I\hat{F}\tau/m} - 1}{q\beta/m} (I\hat{F})\cdot v_0$$

$\hat{F} \Rightarrow$ linear acceleration, $I\hat{F} \Rightarrow$ rotation