February 16, 1999

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 10

SUMMARY

In this lecture we study the application of the STA to electromagnetism. This is one of the most compelling applications of geometric algebra. At various points we will contrast the STA formalism with that of tensors.

- The electromagnetic field strength and observers in relative motion.
- The four Maxwell equations united into one.
- Electromagnetic waves and polarisation states.
- Field energy, the Poynting vector and the stress-energy tensor.
- The field due to a point charge — Coulomb and radiation fields.
- Synchrotron radiation.

Maxwell's Equations

Spacetime vector derivative $\nabla = \gamma^0\partial_t + \gamma^i\partial_i$. Split in the γ_0 frame

$$\nabla\gamma_0 = \partial_t + \gamma^i\gamma_0\partial_i = \partial_t - \boldsymbol{\sigma}_i\partial_i = \partial_t - \boldsymbol{\nabla}$$

Cf $x\gamma_0 = t + \boldsymbol{x}$. Ensures that

$$\nabla x = 4 = \gamma_0\nabla x\gamma_0 = \gamma_0\nabla(t+\boldsymbol{x})$$

The four Maxwell equations are

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \qquad \boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial_t\boldsymbol{B} \qquad \boldsymbol{\nabla}\times\boldsymbol{B} = \boldsymbol{J} + \partial_t\boldsymbol{E}$$

These are Lorentz invariant. Want to make this apparent. Start with source equations and define J with

$$\rho = J\cdot\gamma_0, \qquad \boldsymbol{J} = J\wedge\gamma_0$$

Now form

$$J\gamma_0 = \rho + \boldsymbol{J} = \boldsymbol{\nabla}\cdot\boldsymbol{E} - \partial_t\boldsymbol{E} + \boldsymbol{\nabla}\times\boldsymbol{B}$$

Expect to just use $F = \boldsymbol{E} + I\boldsymbol{B}$ and ∇. Form

$$\nabla\cdot F = (\gamma_0\partial_t + \gamma_0\boldsymbol{\nabla})\times(\boldsymbol{E} + I\boldsymbol{B})$$

Now γ_0 anticommutes with $\boldsymbol{E}$ and commutes with $\boldsymbol{B}$. Get

$$\nabla \cdot F = (-\partial_t \boldsymbol{E} + \boldsymbol{\nabla} \cdot \boldsymbol{E} + \boldsymbol{\nabla} \times \boldsymbol{B})\gamma_0$$

So have $J\gamma_0 = \nabla \cdot F\gamma_0$. Get covariant equation

$$\nabla \cdot F = J$$

The Electromagnetic Field Strength

F is the electromagnetic field strength, or Faraday bivector.

Tensor version is rank-2 antisymmetric tensor $F^{\mu\nu}$

$$F^{\mu\nu} = (\gamma^\nu \wedge \gamma^\mu) \cdot F$$

As a matrix, has components

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

Often see this, but it hides the natural complex structure.

Since $\gamma_0 F \gamma_0 = (-\boldsymbol{E} + I\boldsymbol{B})$, get $\boldsymbol{E}$ and $I\boldsymbol{B}$ from

$$\begin{aligned} \boldsymbol{E} &= \tfrac{1}{2}(F - \gamma_0 F \gamma_0) \\ I\boldsymbol{B} &= \tfrac{1}{2}(F + \gamma_0 F \gamma_0) \end{aligned}$$

Split into $\boldsymbol{E}$ and $I\boldsymbol{B}$ depends on observer velocity (γ_0).

Different observers measure different fields.

Second observer, velocity $v = R\gamma_0\tilde{R}$, comoving frame $\gamma'_\mu = R\gamma_\mu\tilde{R}$. Measures components of electric field

$$E'_i = (\gamma'_i\gamma'_0)\cdot F = (R\boldsymbol{\sigma}_i\tilde{R})\cdot F = \boldsymbol{\sigma}_i\cdot(\tilde{R}FR)$$

Same transformation law as for vectors. Very efficient.

Examples

1. Stationary charges in γ_0 frame set up field

$$F = \boldsymbol{E} = E_x\boldsymbol{\sigma}_1 + E_y\boldsymbol{\sigma}_2$$

Second observer, velocity $\tanh\alpha$ in γ_1 direction, so

$$R = \mathrm{e}^{\alpha\boldsymbol{\sigma}_1/2}$$

Measures the $\boldsymbol{\sigma}_i$ components of

$$\tilde{R}FR = \mathrm{e}^{-\alpha\boldsymbol{\sigma}_1/2}\,F\,\mathrm{e}^{\alpha\boldsymbol{\sigma}_1/2} = E_x\boldsymbol{\sigma}_1 + E_y\,\mathrm{e}^{-\alpha\boldsymbol{\sigma}_1}\,\boldsymbol{\sigma}_2$$

Gives

$$E'_x = E_x, \quad E'_y = \mathrm{ch}(\alpha)E_y, \quad B'_z = -\mathrm{sh}(\alpha)E'_y$$

Much simpler than working with tensors.

2. Construct the scalar + pseudoscalar

$$F^2 = \langle FF\rangle + \langle FF\rangle_4 = \alpha + I\beta$$

But

$$(\tilde{R}FR)(\tilde{R}FR) = \tilde{R}(\alpha + I\beta)R = \alpha + I\beta$$

Both are Lorentz invariant — independent of observer frame.

In γ_0 frame

$$\alpha = \langle(\boldsymbol{E}+I\boldsymbol{B})(\boldsymbol{E}+I\boldsymbol{B})\rangle = \boldsymbol{E}^2 - \boldsymbol{B}^2$$

$$\beta = -\langle I(\boldsymbol{E}+I\boldsymbol{B})(\boldsymbol{E}+I\boldsymbol{B})\rangle = 2\boldsymbol{E}\cdot\boldsymbol{B}$$

First is Lagrangian density. Second less common.

The Remaining Equations

Remaining Maxwell equations are

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \qquad \boldsymbol{\nabla}\times\boldsymbol{E} = -\partial_t\boldsymbol{B}$$

First can be written

$$0 = \boldsymbol{\nabla}\wedge(I\boldsymbol{B}) = \nabla\wedge(I\boldsymbol{B})\wedge\gamma_0 = \nabla\wedge F\wedge\gamma_0$$

So form

$$\begin{aligned}\nabla\wedge F &= (\gamma_0\partial_t + \gamma_0\boldsymbol{\nabla})\wedge(\boldsymbol{E}+I\boldsymbol{B}) \\ &= (I\partial_t\boldsymbol{B} + \boldsymbol{\nabla}\wedge\boldsymbol{E} + I\boldsymbol{\nabla}\cdot\boldsymbol{B})\gamma_0 \\ &= (\partial_t\boldsymbol{B} + \boldsymbol{\nabla}\times\boldsymbol{E} + \boldsymbol{\nabla}\cdot\boldsymbol{B})I\gamma_0\end{aligned}$$

Term in bracket vanishes, get second covariant equation

$$\nabla\wedge F = 0$$

Now have $\nabla \cdot F = J$ and $\nabla \wedge F = 0$. In tensors get two equations

$$\partial_\mu F^{\mu\nu} = J^\nu, \qquad \partial_\mu \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = 0$$

Same for differential forms. But we can simplify further. Combine into a geometric product

$$\nabla F = J$$

All of Maxwell's equations in one! Big advantage — ∇ is invertible.

- Develop first-order diffraction theory.
- Better numerics.
- Propagation easy to understand.

The Vector Potential

Introduce vector potential A so that

$$F = \nabla \wedge A, \qquad \Rightarrow \nabla \wedge F = \nabla \wedge (\nabla \wedge A) = 0$$

Can add gradient of a scalar to A (a gauge freedom). Usually choose Lorentz gauge,

$$\nabla \cdot A = 0, \qquad F = \nabla A$$

Get wave equation

$$\nabla F = \nabla^2 A = J$$

Electromagnetic Waves

Free-field equation $\nabla F = 0$. Try

$$F = F_0 \, \mathrm{e}^{I k \cdot x}$$

Equation reduces to

$$k F_0 = 0$$

$\times$ by k, $\Rightarrow$ $k^2 = 0$. F_0 contains factor of k, have

$$F_0 = k \wedge n = kn, \qquad k \cdot n = 0$$

Add multiple of k to n. Remove component in spacetime plane of k.

Example

Wave travelling in $+z$ direction, frequency ω

$$k = \omega(\gamma_0 + \gamma_3)$$

n contains γ_1, γ_2 only so

$$\begin{aligned} F &= -(\gamma_0 + \gamma_3)(\alpha\gamma_1 + \beta\gamma_2) \, \mathrm{e}^{I\omega(t - z)} \\ &= (1 + \boldsymbol{\sigma}_3)(\alpha\boldsymbol{\sigma}_1 + \beta\boldsymbol{\sigma}_2) \, \mathrm{e}^{I\omega(t - z)} \end{aligned}$$

Multivector $1 + \boldsymbol{\sigma}_3$ satisfies

$$\boldsymbol{\sigma}_3(1 + \boldsymbol{\sigma}_3) = 1 + \boldsymbol{\sigma}_3, \qquad (1 + \boldsymbol{\sigma}_3)^2 = 2(1 + \boldsymbol{\sigma}_3)$$

Convert phase factor I to rotations in $I\boldsymbol{\sigma}_3$ plane

$$
\begin{aligned}
(1+\boldsymbol{\sigma}_3)\,\mathrm{e}^{I\phi} &= (1+\boldsymbol{\sigma}_3)[\cos(\phi) + I\sin(\phi)] \\
&= (1+\boldsymbol{\sigma}_3)[\cos(\phi) + I\boldsymbol{\sigma}_3\sin(\phi)] = (1+\boldsymbol{\sigma}_3)\,\mathrm{e}^{I\boldsymbol{\sigma}_3\phi}
\end{aligned}
$$

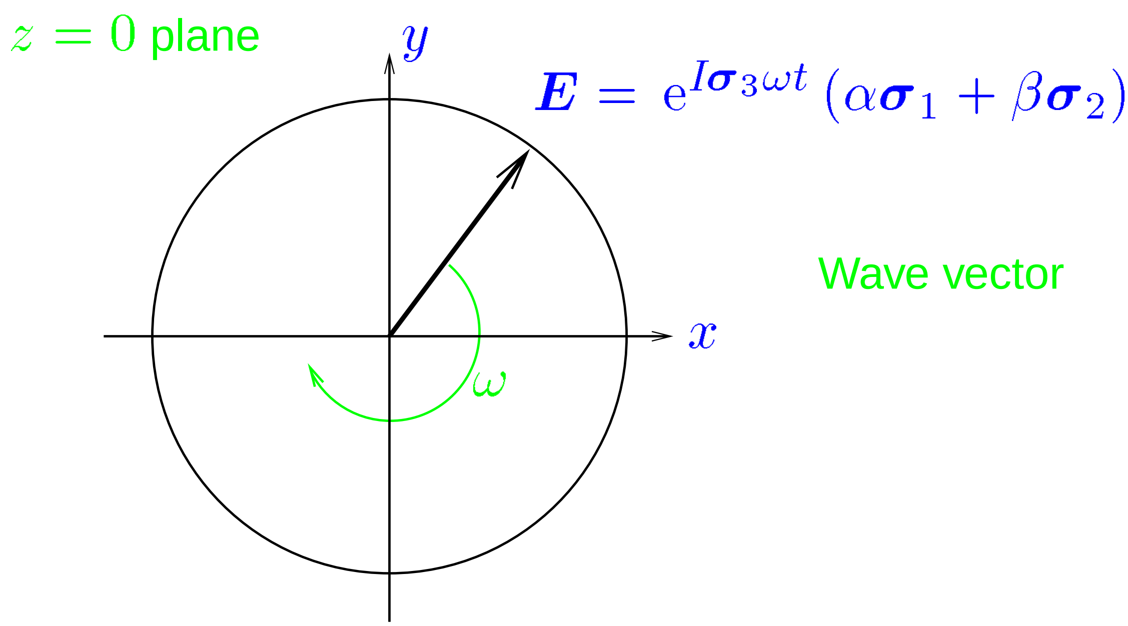
Now have

$$F = \mathrm{e}^{I\boldsymbol{\sigma}_3\omega(t-z)}\,(1+\boldsymbol{\sigma}_3)(\alpha\boldsymbol{\sigma}_1 + \beta\boldsymbol{\sigma}_2)$$

Extract

$$
\begin{aligned}
\boldsymbol{E} &= \mathrm{e}^{I\boldsymbol{\sigma}_3\omega(t-z)}\,(\alpha\boldsymbol{\sigma}_1 + \beta\boldsymbol{\sigma}_2) \\
\boldsymbol{B} &= \mathrm{e}^{I\boldsymbol{\sigma}_3\omega(t-z)}\,(-\beta\boldsymbol{\sigma}_1 + \alpha\boldsymbol{\sigma}_2)
\end{aligned}
$$

$\boldsymbol{E}$ rotates clockwise in constant z plane. This is left-hand circularly polarised light.



Change sign of exponent for right-handed.

Can also write

$$(1+\boldsymbol{\sigma}_3)(\alpha\boldsymbol{\sigma}_1+\beta\boldsymbol{\sigma}_2)=(1+\boldsymbol{\sigma}_3)\boldsymbol{\sigma}_1(\alpha-\beta I)$$

General decomposition into circularly polarised modes

$$F=(1+\boldsymbol{\sigma}_3)\boldsymbol{\sigma}_1[R\,\mathrm{e}^{I\omega(z-t)}+L\,\mathrm{e}^{-I\omega(z-t)}]$$

R and L are 'complex' (scalar + pseudoscalar) coefficients. Build plane and elliptic polarisations from these. *Eg.* get linearly polarised with $R=L=1/2$,

$$F=(1+\boldsymbol{\sigma}_3)\boldsymbol{\sigma}_1\cos(\omega t-\omega z).$$

Field Energy and Momentum

The field energy is

$$\mathcal{E}=\tfrac{1}{2}(\boldsymbol{E}^2+\boldsymbol{B}^2)$$

and momentum given by Poynting vector

$$\boldsymbol{P}=\boldsymbol{E}\times\boldsymbol{B}=-\boldsymbol{E}\cdot(I\boldsymbol{B})$$

Combine into spacetime vector

$$\begin{aligned}P &= \tfrac{1}{2}(\boldsymbol{E}^2+\boldsymbol{B}^2)\gamma_0+\tfrac{1}{2}(I\boldsymbol{B}\boldsymbol{E}-\boldsymbol{E}I\boldsymbol{B})\gamma_0\\ &= \tfrac{1}{2}(\boldsymbol{E}+I\boldsymbol{B})(\boldsymbol{E}-I\boldsymbol{B})\gamma_0\\ &= \tfrac{1}{2}F(-\gamma_0F\gamma_0)\gamma_0=-\tfrac{1}{2}F\gamma_0F\end{aligned}$$

Have constructed the stress-energy tensor. We write this

$$\mathsf{T}(a) = -\frac{1}{2} F a F$$

Returns the flux of 4-momentum across the hypersurface perpendicular to a. Fundamental to relativistic field theory.

PROPERTIES

1. The stress-energy tensor is symmetric (usually):

$$a \cdot \mathsf{T}(b) = -\frac{1}{2}\langle aFbF \rangle = -\frac{1}{2}\langle FaFb \rangle = \mathsf{T}(a) \cdot b$$

2. Energy density $v \cdot \mathsf{T}(v) > 0$ for all future-pointing v. Otherwise matter is exotic

3. Total flux over closed hypersurface is zero (no sources or sinks). Requires

$$\int_{\partial V} dA\, \mathsf{T}(n) = 0 = \int_V \dot{\mathsf{T}}(\dot{\nabla})\, dV$$

True for any hypersurface, so $\dot{\mathsf{T}}(\dot{\nabla}) = 0$. Or use symmetry of $\mathsf{T}(a)$

$$\nabla \cdot \mathsf{T}(a) = 0 \qquad \forall \text{ const } a$$

Proof for free-field electromagnetism:

$$\dot{\mathsf{T}}(\dot{\nabla}) = -\frac{1}{2}[\dot{F}\dot{\nabla}F + F\nabla F] = 0$$

4. Can define conserved vector

$$P_{\text{tot}} = \int d^3x \, \mathsf{T}(\gamma_0)$$

This is independent of frame — a covariant vector.

5. With sources get flow of energy:

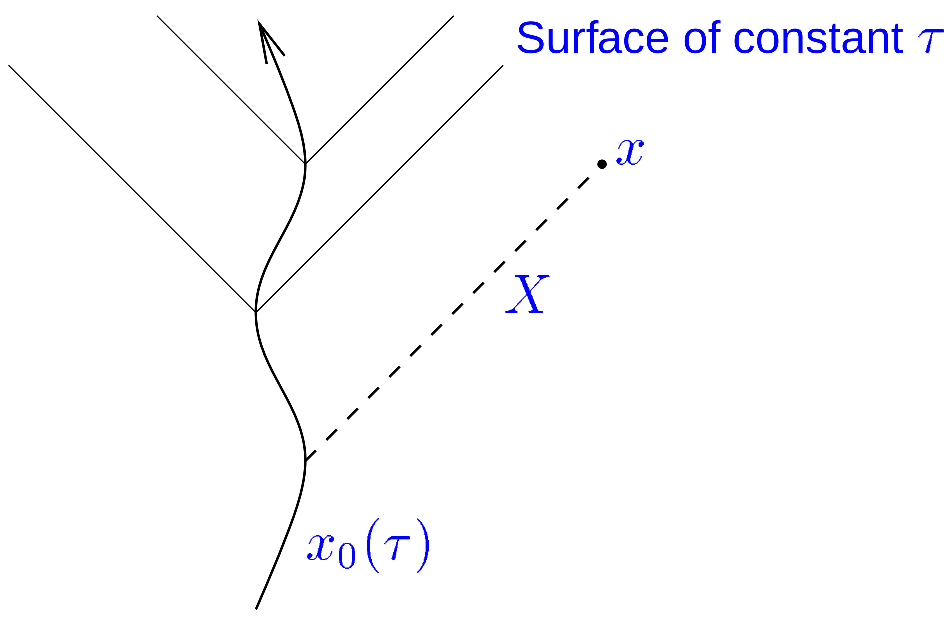
$$\dot{\mathsf{T}}(\dot{\nabla}) = -\tfrac{1}{2}(-JF + FJ) = J \cdot F$$

In γ_0 frame get

$$J \cdot F = -[\boldsymbol{J} \cdot \boldsymbol{E} + \rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}]\gamma_0$$

First term is work. Second recovers the Lorentz force law.

Point Charges



Point charge q, world-line $x_0(\tau)$. Observer at x. Influence from intersection of past light-cone with charge's worldline.

Define

$$X \equiv x - x_0(\tau), \quad X^2 = 0$$

View τ as a field, value extended over the forward light cone.

Need Liénard-Wiechert potential,

$$A = \frac{q}{4\pi\epsilon_0}\frac{v}{X\cdot v},$$

where $v = \dot{x}_0$. Now differentiate $X^2 = 0$

$$\begin{aligned} \dot{\nabla}\,\dot{X}\cdot X &= \dot{\nabla}\,\dot{x}\cdot X - \nabla\tau(\partial_\tau x_0)\cdot X \\ &= X - \nabla\tau(v\cdot X) = 0 \end{aligned}$$

So

$$\nabla\tau = \frac{X}{X\cdot v}$$

Gradient of τ points in direction of constant τ! A peculiarity of null surfaces. Confirmed that τ is an adjunct field.

Next need

$$\nabla(X\cdot v) = \dot{\nabla}\,\dot{X}\cdot v + \nabla\tau\, X\cdot(\partial_\tau v) = v - \nabla\tau + \nabla\tau\, X\cdot\dot{v}$$

where $\dot{v} = \partial_\tau v$. (Do not confuse with overdots for scope).

Now get

$$\nabla A = \frac{q}{4\pi\epsilon_0}\left(\frac{\nabla v}{X\cdot v} - \frac{1}{(X\cdot v)^2}\nabla(X\cdot v)v\right)$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{X\dot{v}}{(X\cdot v)^2} - \frac{1}{(X\cdot v)^2} - \frac{(X\,X\cdot\dot{v} - X)v}{(X\cdot v)^3}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{X\wedge\dot{v}}{(X\cdot v)^2} + \frac{X\wedge v - X\cdot\dot{v}\,X\wedge v}{(X\cdot v)^3}\right)$$

A pure bivector so A in Lorentz gauge. Now write

$$X\cdot v\,X\wedge\dot{v} - X\cdot\dot{v}\,X\wedge v = -X\wedge[X\cdot(\dot{v}\wedge v)] = \tfrac{1}{2}X\dot{v}\wedge vX$$

Define acceleration bivector $\Omega_v = \dot{v}\wedge v$, get

$$F = \frac{q}{4\pi\epsilon_0}\frac{X\wedge v + \frac{1}{2}X\Omega_v X}{(X\cdot v)^3}.$$

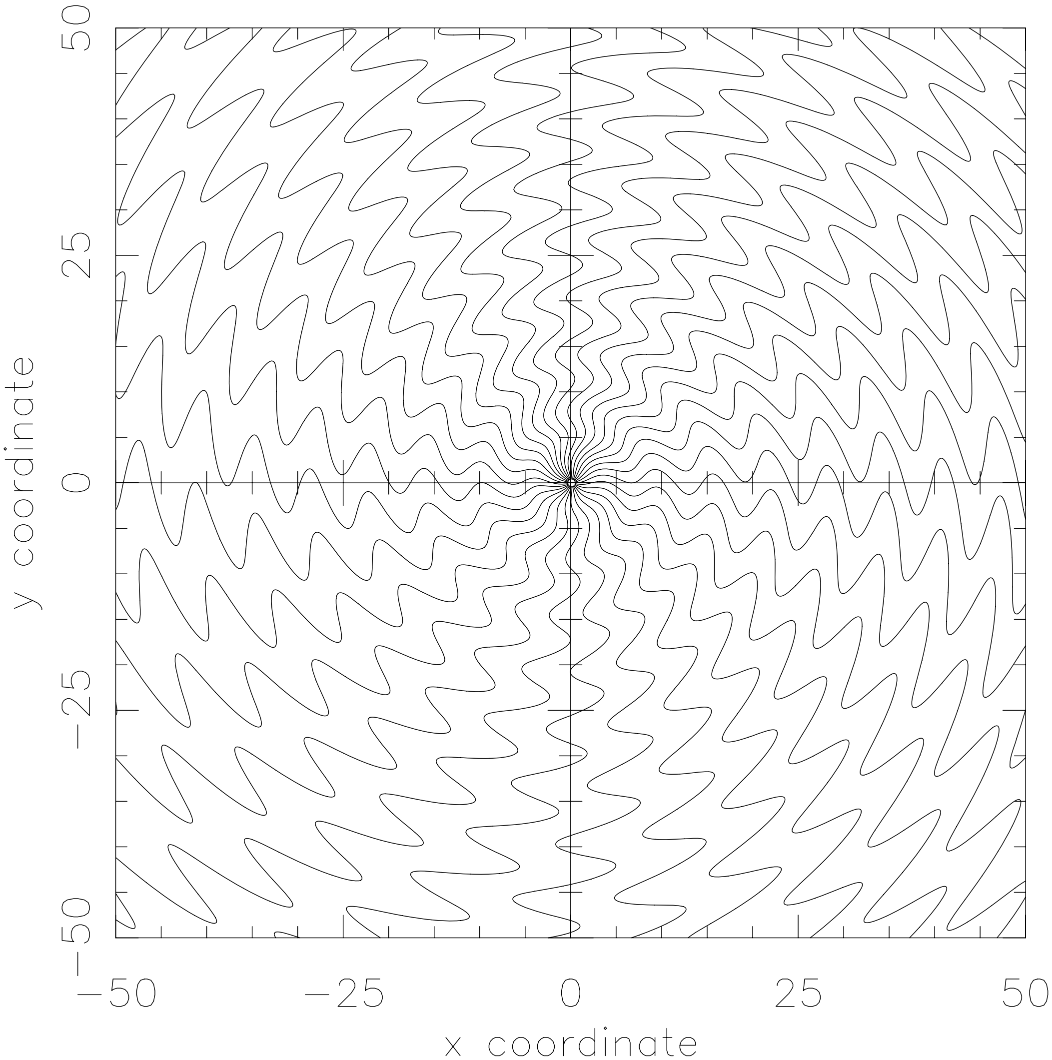
First term is Coulomb field in rest frame. Second is radiation term,

$$F_{rad} = \frac{q}{4\pi\epsilon_0}\frac{\frac{1}{2}X\Omega_v X}{(X\cdot v)^3},$$

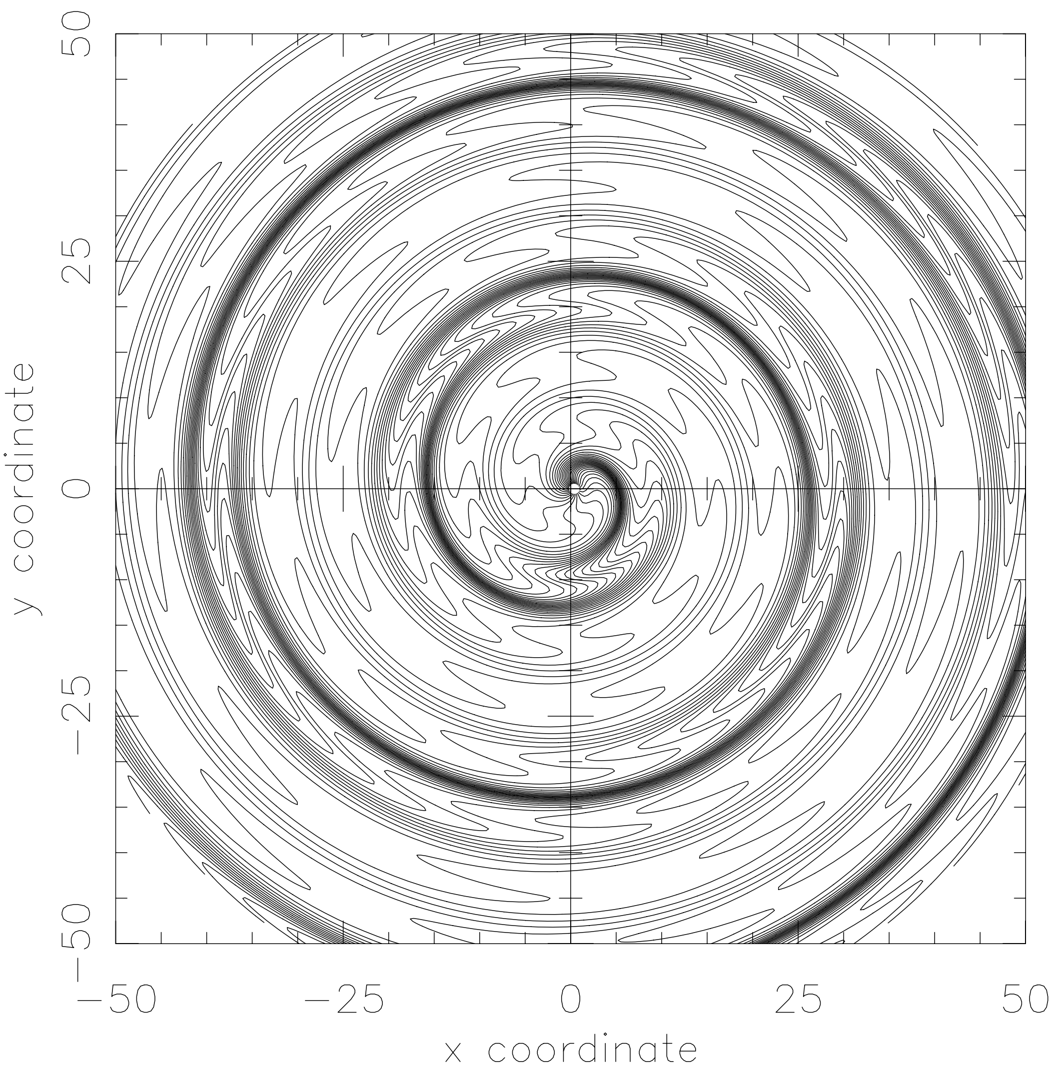
The rest-frame acceleration projected down the null-vector X.

Example — Circular Orbits

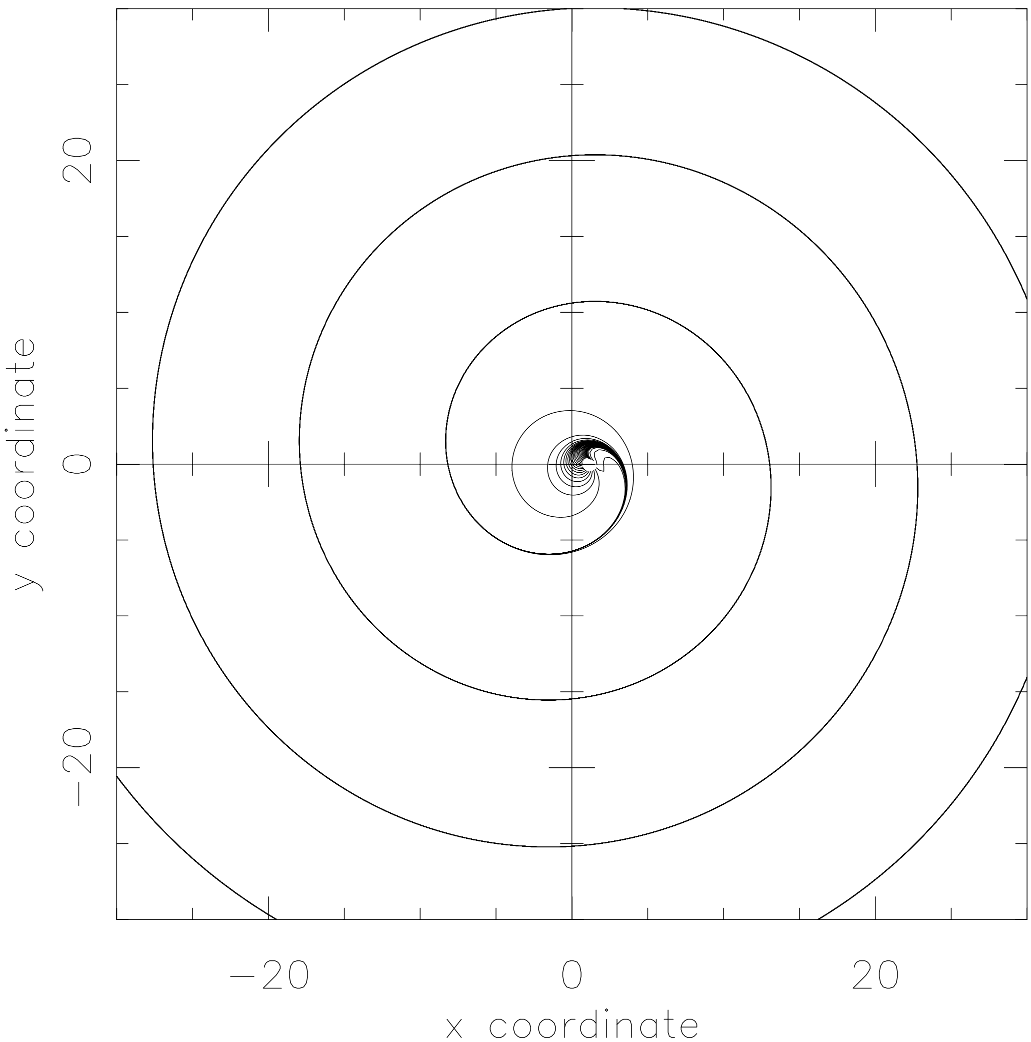
Use circular orbit from description of Thomas precession. Explicit formulae for v and Ω_v. Find τ for each numerically. Plot field lines for various values of the angular velocity.



$\alpha = 0.1$, velocity $\tanh(\alpha)$ is low. Get gentle wavy pattern.



Intermediate velocities, $\alpha = 0.4$. Complicated structure emerging. Field lines start to concentrate together.



Synchrotron Radiation. By $\alpha = 1$ field lines concentrate into pure synchrotron pulses. Radiation focussed in charge's direction of motion.