PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 11

SUMMARY

In this lecture we will study the application of the STA to quantum physics, focusing attention on quantum spin. The Pauli and Dirac matrix algebras are Clifford algebras, so quantum spin has a natural expression in the STA. But this has some surprising consequences . . .

- Non-relativistic quantum spin. Pauli matrices and spinors.
- Spinors in the STA, rotors and observables.
- Particle in a magnetic field. The quantum Hamiltonian and its STA form.
- Magnetic Resonance Imaging.
- Relativistic quantum spin, Dirac matrices and spinors, and spacetime observables.

NON-RELATIVISTIC QUANTUM SPIN

Pauli matrices are

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Matrix operators (with hats). Not elements of a geometric algebra, though satisfy the same relations

The $\{\hat{\sigma}_k\}$ act on 2-component Pauli spinors

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

ψ_1, ψ_2 complex. (Use bras and kets to distinguish from multivectors.)

$|\psi\rangle$ in two-dimensional complex vector space. Seek multivector equivalent. Form matrix

$$\begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix} = \begin{pmatrix} \psi_1 & \psi_3 \\ \psi_2 & \psi_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

(ψ_3 and ψ_4 irrelevant coefficients.)

1 to 1 map between 2×2 complex matrices and multivectors: Decompose into Pauli basis, then $\hat{\sigma}_k \mapsto \boldsymbol{\sigma}_k$ and $i \mapsto I$. Get multivector equivalent

$$\psi \tfrac{1}{2}(1 + \boldsymbol{\sigma}_3)$$

But factor on right is a projection operator

$$\tfrac{1}{2}(1+\boldsymbol{\sigma}_3) = \boldsymbol{\sigma}_3 \tfrac{1}{2}(1+\boldsymbol{\sigma}_3)$$

$\implies$ keep ψ in even subalgebra. This is 4-dimensional.

Now strip off projector. Go through for each term. Establishes map

$$|\psi\rangle = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{pmatrix} \quad \leftrightarrow \quad \psi = a^0 + a^k I \boldsymbol{\sigma}_k.$$

For spin-up $|+\rangle$, and spin-down $|-\rangle$ get

$$|+\rangle \quad \leftrightarrow \quad 1 \qquad |-\rangle \quad \leftrightarrow \quad -I\boldsymbol{\sigma}_2$$

(Details of preceding process largely irrelevant — just a means of finding the correct map.)

Pauli Operators

Action of the quantum operators $\{\hat{\sigma}_k\}$ on states $|\psi\rangle$ has an analogous operation on the multivector ψ:

$$\hat{\sigma}_k|\psi\rangle \quad \leftrightarrow \quad \boldsymbol{\sigma}_k \psi \boldsymbol{\sigma}_3 \quad (k = 1, 2, 3).$$

$\boldsymbol{\sigma}_3$ on the right-hand side is a remnant of the $\frac{1}{2}(1+\boldsymbol{\sigma}_3)$ projector

—ensures that $\boldsymbol{\sigma}_k \psi \boldsymbol{\sigma}_3$ stays in the even subalgebra. Verify that the translation procedure is consistent by computation;

e.g.

$$\hat{\sigma}_1|\psi\rangle = \begin{pmatrix} -a^2 + ia^1 \\ a^0 + ia^3 \end{pmatrix}$$

translates to

$$-a^2 + a^1 I\boldsymbol{\sigma}_3 - a^0 I\boldsymbol{\sigma}_2 + a^3 I\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_1 \psi \boldsymbol{\sigma}_3.$$

Also need translation for multiplication by the unit imaginary i. Do this via noting

$$\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}.$$

See multiplication of both components of $|\psi\rangle$ achieved by multiplying by the product of the three matrix operators.

Therefore arrive at the translation

$$i|\psi\rangle \quad \leftrightarrow \quad \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3\psi(\boldsymbol{\sigma}_3)^3 = \psi I\boldsymbol{\sigma}_3.$$

Unit imaginary of quantum theory is replaced by right multiplication by the bivector $I\boldsymbol{\sigma}_3$.

This is very suggestive (though fact it is $I\boldsymbol{\sigma}_3$ is a feature of our chosen representation).

PAULI OBSERVABLES

Next need to establish the quantum inner product for our multivector forms of spinors.

Before this, we first introduce the Hermitian adjoint as

$$M^\dagger = \gamma_0 \tilde{M} \gamma_0.$$

For the $\boldsymbol{\sigma}_i$ we find that $\boldsymbol{\sigma}_i^\dagger = \boldsymbol{\sigma}_i$, whereas $I^\dagger = -I$.

Thus the dagger operation is equivalent to reversion in 3-d.

Therefore employ the dagger symbol for the operation of 3-d reversion and reserve the tilde symbol for the spacetime reverse.

(Work on Pauli spinors then sits naturally in the full STA — note however, Hermitian conjugation is frame dependent).

First consider the real part of the quantum inner product.

Have

$$\Re\langle\psi|\phi\rangle = \Re(\psi_1^\dagger \phi_1 + \psi_2^\dagger \phi_2).$$

This is reproduced by

$$\Re\langle\psi|\phi\rangle \quad \leftrightarrow \quad \langle\psi^\dagger \phi\rangle,$$

so that, for example, $\langle\psi|\psi\rangle$ translates to

$$\langle\psi^\dagger \psi\rangle = \langle(a^0 - a^j I\boldsymbol{\sigma}_j)(a^0 + a^k I\boldsymbol{\sigma}_k)\rangle = (a^0)^2 + a^k a^k.$$

(Note that no spatial integral is implied in our use of the bra-ket notation.) Since

$$\langle\psi|\phi\rangle = \Re\langle\psi|\phi\rangle - i\Re\langle\psi|i\phi\rangle,$$

the full inner product can be written

$$\langle\psi|\phi\rangle \quad \leftrightarrow \quad \langle\psi^\dagger\phi\rangle - \langle\psi^\dagger\phi I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3.$$

Right hand side projects out the 1 and $I\boldsymbol{\sigma}_3$ components from the geometric product $\psi^\dagger\phi$.

Result is written $\langle A\rangle_q$. For even grade multivectors in 3-d this projection has the simple form

$$\langle A\rangle_q = \tfrac{1}{2}(A + \boldsymbol{\sigma}_3 A\boldsymbol{\sigma}_3).$$

The Spin Vector

Now consider the expectation value of the spin in the k-direction,

$$\langle\psi|\hat{\sigma}_k|\psi\rangle \quad \leftrightarrow \quad \langle\psi^\dagger\boldsymbol{\sigma}_k\psi\boldsymbol{\sigma}_3\rangle - \langle\psi^\dagger\boldsymbol{\sigma}_k\psi I\rangle I\boldsymbol{\sigma}_3.$$

$\psi^\dagger I\boldsymbol{\sigma}_k\psi$ reverses to give minus itself, so has zero scalar part.

Also note that in 3-d $\psi\boldsymbol{\sigma}_3\psi^\dagger$ is both odd grade and reverses to itself, so is a pure vector.

Therefore define the spin vector

$$\boldsymbol{s} \equiv \psi\boldsymbol{\sigma}_3\psi^\dagger.$$

The quantum expectation now reduces to

$$\langle\psi|\hat{\sigma}_k|\psi\rangle = \boldsymbol{\sigma}_k \cdot \boldsymbol{s}.$$

Note this new expression has a rather different interpretation to that usually encountered in quantum theory.

Rather than forming the expectation value of a quantum operator, we project out the kth component of the vector $\boldsymbol{s}$.

SPINORS AND ROTATIONS

The STA approach focuses attention on the vector $\boldsymbol{s}$, whereas the operator/matrix theory treats only its individual components.

Now define the scalar

$$\rho \equiv \psi\psi^\dagger.$$

The spinor ψ then decomposes into

$$\psi = \rho^{1/2} R,$$

where $R = \rho^{-1/2}\psi$. The multivector R satisfies $RR^\dagger = 1$, so is a rotor. In this approach, Pauli spinors are simply unnormalised rotors!

The spin-vector $\boldsymbol{s}$ can now be written as

$$\boldsymbol{s} = \rho R \boldsymbol{\sigma}_3 R^\dagger.$$

The double-sided construction of the expectation value contains an instruction to rotate the fixed $\boldsymbol{\sigma}_3$ axis into the spin direction and dilate it.

This view offers a number of insights.

E.g. suppose that the vector $\boldsymbol{s}$ is to be rotated to a new vector $R_0 \boldsymbol{s} R_0^\dagger$. The rotor group combination law tells us that R transforms to $R_0 R$.

This induces the spinor transformation law

$$\psi \mapsto R_0 \psi.$$

This explains the 'spin-1/2' nature of spinor wave functions.

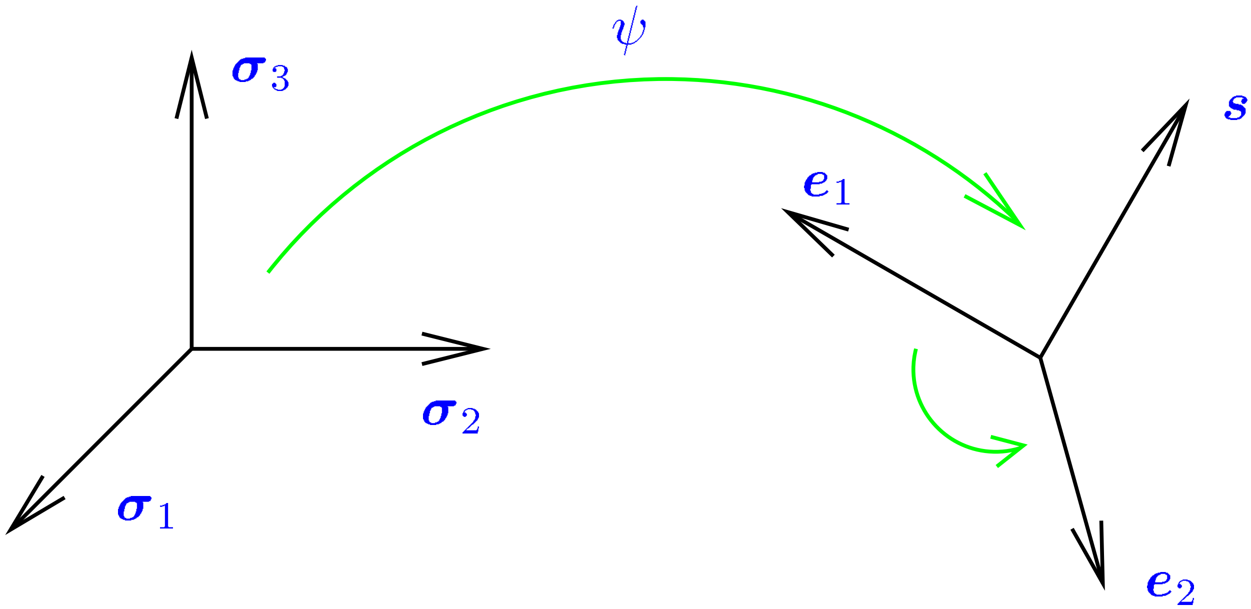

Figure 1: *The Spin Vector*. The normalised spinor ψ transforms the initial, reference frame onto the frame $\{e_k\}$. The vector e_3 is the spin vector. A phase transformation of ψ generates a rotation in the $e_1 e_2$ plane.

Note in writing the spin vector as $s = \psi \sigma_3 \psi^\dagger$ we are not singling out some preferred direction in space.

The σ_3 on the right of ψ represents a vector in a 'reference' frame. All physical vectors, like s, are obtained by rotating this frame onto the physical value.

There is nothing special about σ_3 — one can choose any (constant) reference frame and use the appropriate rotation onto s, in the same way that there is nothing special about the orientation of the reference configuration of a rigid body.

Application — Magnetic Fields

Particles with non-zero spin also have a magnetic moment. This is conventionally expressed as the operator relation

$$\hat{\mu}_k = \gamma \hat{s}_k,$$

where $\hat{\mu}_k$ is the magnetic moment operator, γ is the gyromagnetic ratio and $\hat{s}_k$ is the spin operator. The gyromagnetic ratio is usually written in the form

$$\gamma = g \frac{-q}{2m},$$

where m is the particle mass, q is the charge and g is the reduced gyromagnetic ratio. The latter are determined experimentally to be

electron	$g_e = 2$	(actually $2(1 + \alpha/2\pi +)$)
proton	$g_p = 5.587$	
neutron	$g_n = -3.826$	

All of the above are spin-$1/2$ particles and conventionally write

$$\hat{s}_k = \tfrac{1}{2}\hbar\hat{\sigma}_k.$$

The $\hat{\sigma}_k$ matrix operators are then viewed as the components of a single vector $\hat{\boldsymbol{\sigma}}$.

Particle in a Magnetic Field

Now suppose that the particle is in a magnetic field. We introduce the Hamiltonian operator

$$\hat{H} = -\tfrac{1}{2}\gamma\hbar B_k\hat{\sigma}_k = -\hat{\boldsymbol{\mu}}\cdot\boldsymbol{B}.$$

The spin state at time t is then written as

$$|\psi(t)\rangle = a_1(t)|+\rangle + a_2(t)|-\rangle,$$

with a_1 and a_2 general complex coefficients. The dynamical equation for these coefficients is given by the time-dependent Schrödinger equation

$$\hat{H}|\psi\rangle = i\hbar\frac{d|\psi\rangle}{dt}.$$

This is conventionally hard to analyse, because one ends up with a pair of coupled differential equations in a_1 and a_2.

Let's see what the Schrödinger equation looks like in our new setup. We first write the equation in the form

$$\frac{d|\psi\rangle}{dt} = \tfrac{1}{2}\gamma i B_k\hat{\sigma}_k|\psi\rangle.$$

Replacing $|\psi\rangle$ by the multivector ψ the left-hand side is simply $\dot{\psi}$ (the dot denotes the time derivative).

The right-hand side involves multiplication of the spinor $|\psi\rangle$ by

$i\hat{\sigma}_k$, so replace by

$$i\hat{\sigma}_k|\psi\rangle \quad \leftrightarrow \quad \boldsymbol{\sigma}_k\psi\boldsymbol{\sigma}_3(I\boldsymbol{\sigma}_3) = I\boldsymbol{\sigma}_k\psi.$$

Our STA version of the Schrödinger is therefore simply

$$\dot{\psi} = \frac{1}{2}\gamma B_k I\boldsymbol{\sigma}_k\psi = \frac{1}{2}\gamma I\boldsymbol{B}\psi.$$

If we now decompose ψ into $\rho^{1/2}R$ we see that

$$\dot{\psi}\tilde{\psi} = \frac{1}{2}\dot{\rho}\rho + \rho\dot{R}\tilde{R} = \frac{1}{2}\rho\gamma I\boldsymbol{B}.$$

The right-hand side is a bivector, so ρ must be constant and the dynamics reduces to

$$\dot{R} = \frac{1}{2}\gamma I\boldsymbol{B}R.$$

The quantum theory of a spin-$1/2$ particle in a magnetic field reduces to another rotor equation!

Recovering a rotor equation explains the difficulty of the traditional analysis based on a pair of coupled equations for the components of $|\psi\rangle$.

Latter fails to capture the fact that there is a rotor underlying the dynamics, and so carries along redundant degrees of freedom in the normalisation.

Also, the separation of a rotor into a pair of components is far from natural.

As a simple example, consider a constant field $\boldsymbol{B} = B_0\boldsymbol{\sigma}_3$.

The rotor equation integrates immediately to give

$$\psi(t) = \mathrm{e}^{\gamma B_0 t I \boldsymbol{\sigma}_3/2} \psi_0.$$

The spin vector $\boldsymbol{s}$ therefore just precesses about the 3 axis at a rate $\omega_0 = \gamma B_0$.

Traditional methods are much more complicated!

Magnetic Resonance Imaging

More interesting example is to include an oscillatory $\boldsymbol{B}$ field $(B_1 \cos(\omega t), B_1 \sin(\omega t), 0)$ together with a constant field along the z-axis.

This oscillatory field induces transitions (spin-flips) between the up and down states.

Interesting system of great practical importance. (Basis of magnetic resonance imaging and Rabi molecular beam spectroscopy.)

To study this system we first write the $\boldsymbol{B}$ field as

$$\begin{aligned} & B_1(\cos(\omega t)\boldsymbol{\sigma}_1 + \sin(\omega t)\boldsymbol{\sigma}_2) + B_0\boldsymbol{\sigma}_3 \\ &= \mathrm{e}^{-\omega t I \boldsymbol{\sigma}_3} B_1 \boldsymbol{\sigma}_1 + B_0 \boldsymbol{\sigma}_3 \\ &= \mathrm{e}^{-\omega t I \boldsymbol{\sigma}_3/2} (B_1 \boldsymbol{\sigma}_1 + B_0 \boldsymbol{\sigma}_3) \mathrm{e}^{\omega t I \boldsymbol{\sigma}_3/2} \end{aligned}$$

Now define

$$S = \mathrm{e}^{-\omega t I \boldsymbol{\sigma}_3/2} \quad \text{and} \quad \boldsymbol{B}_c = B_1 \boldsymbol{\sigma}_1 + B_0 \boldsymbol{\sigma}_3$$

so that we can write $\boldsymbol{B} = S\boldsymbol{B}_c\tilde{S}$. The rotor equation can now be written

$$\tilde{S}\dot{\psi} = \tfrac{1}{2}\gamma I\boldsymbol{B}_c\tilde{S}\psi,$$

where we have pre-multiplied by $\tilde{S}$. Now noting that

$$\dot{\tilde{S}} = \tfrac{1}{2}\omega I\boldsymbol{\sigma}_3\tilde{S}$$

we see that

$$\frac{d}{dt}(\tilde{S}\psi) = \tfrac{1}{2}(\gamma I\boldsymbol{B}_c + \omega I\boldsymbol{\sigma}_3)\tilde{S}\psi.$$

It is now $\tilde{S}\psi$ that satisfies a rotor equation with a constant field. The solution is straightforward,

$$\tilde{S}\psi(t) = \exp\left(\frac{t}{2}(\gamma I\boldsymbol{B}_c + \omega I\boldsymbol{\sigma}_3)\right)\psi_0,$$

and we arrive at

$$\psi(t) = \exp\left(-\frac{wt}{2}I\boldsymbol{\sigma}_3\right)\exp\left(\frac{t}{2}[(\omega_0+\omega)I\boldsymbol{\sigma}_3 + \omega_1 I\boldsymbol{\sigma}_1]\right)\psi_0$$

where $\omega_1 = \gamma B_1$. There are three separate frequencies in this solution, which contains a wealth of interesting physics. Needless to say, this derivation is a vast improvement over standard methods!

To complete our analysis we must relate our solution to the results of experiments. Suppose that at time $t = 0$ we switch on the oscillating field. The particle is initially in a spin-up state, so $\psi_0 = 1$, which also ensures that the state is normalised. The probability that at time t the particle is in the spin-down state is

$$P_- = |\langle - | \psi(t) \rangle|^2$$

We therefore need to form the inner product

$$\langle - | \psi(t) \rangle \quad \leftrightarrow \quad \langle I\boldsymbol{\sigma}_2 \psi \rangle_q = \langle I\boldsymbol{\sigma}_2 \psi \rangle - I\boldsymbol{\sigma}_3 \langle I\boldsymbol{\sigma}_2 \psi I\boldsymbol{\sigma}_3 \rangle$$
$$= \langle I\boldsymbol{\sigma}_2 \psi \rangle - I\boldsymbol{\sigma}_3 \langle I\boldsymbol{\sigma}_1 \psi \rangle.$$

To find this inner product we write

$$\psi(t) = \mathrm{e}^{-wtI\boldsymbol{\sigma}_3/2} \left[\cos(\alpha t/2) + I\hat{B} \sin(\alpha t/2) \right]$$

where

$$\hat{B} = \frac{(\omega_0 + \omega)\boldsymbol{\sigma}_3 + \omega_1 \boldsymbol{\sigma}_1}{\alpha}, \quad \alpha = \sqrt{(w + w_0)^2 + \omega_1^2}.$$

The only term giving a contribution in the $I\boldsymbol{\sigma}_1$ and $I\boldsymbol{\sigma}_2$ planes is that in $\omega_1 I\boldsymbol{\sigma}_1/\alpha$. We therefore have

$$\langle I\boldsymbol{\sigma}_2 \psi \rangle_q = \frac{\omega_1 \sin(\alpha t/2)}{\alpha} \, \mathrm{e}^{-wtI\boldsymbol{\sigma}_3/2} \, I\boldsymbol{\sigma}_3$$

and the probability is immediately

$$P_- = \left(\frac{\omega_1 \sin(\alpha t/2)}{\alpha} \right)^2 .$$

The maximum value is at $\alpha t = \pi$, and the probability at this time is maximised by choosing α as small as possible. This is achieved by setting $\omega = -\omega_0 = -\gamma B_0$. This is the spin resonance condition.

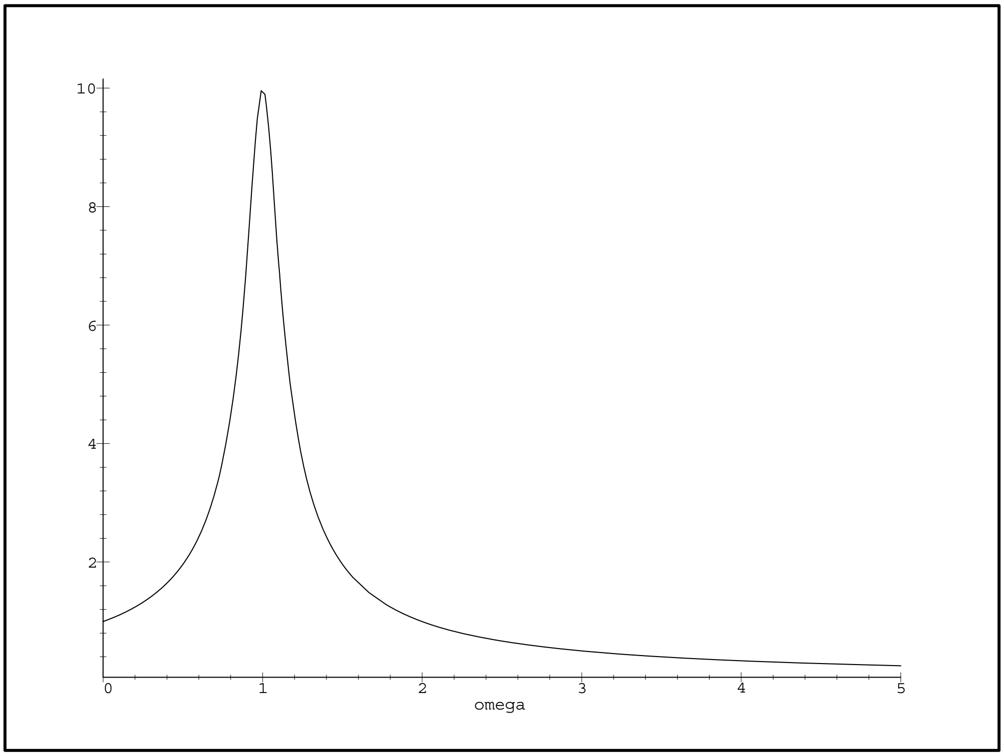

Figure 2: Example curve of $1/\alpha$ versus ω.