PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 12

SUMMARY

In this lecture we will study the application of the STA to relativistic quantum theory. The key to this is the Dirac theory, based on the algebra of the γ-matrices. These form a representation of the STA. Using a similar device to that for Pauli spinors, Dirac spinors also sit naturally in the STA.

- Relativistic quantum spin. Dirac matrices and spinors, and their STA equivalent.
- Rotors, spacetime observables, the current and spin.
- Point particle in a magnetic field and the gyromagnetic ratio.
- The Dirac equation. Massless and massive forms.
- Plane-wave states. Another appliction for pure boosts.
- Hamiltonian form and angular operators.
- Central potentials and 3-d analytic functions
- The Hydrogen atom

Relativistic Quantum Spin

Relativistic quantum mechanics of spin-$1/2$ particles described by Dirac theory. The Dirac matrix operators are

$$\hat{\gamma}_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \hat{\gamma}_k = \begin{pmatrix} 0 & -\hat{\sigma}_k \\ \hat{\sigma}_k & 0 \end{pmatrix}, \hat{\gamma}_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

where $\hat{\gamma}_5 = -i\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3$ and I is the 2×2 identity matrix.

Act on Dirac spinors. 4 complex components (8 real degrees of freedom).

Follow same procedure as Pauli case. Map spinors onto elements of the 8-dimensional even subalgebra of the STA. First write

$$|\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\eta\rangle \end{pmatrix},$$

where $|\phi\rangle$ and $|\eta\rangle$ are 2-component spinors. Know how to represent the latter. Full map is simply

$$|\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\eta\rangle \end{pmatrix} \leftrightarrow \psi = \phi + \eta\boldsymbol{\sigma}_3$$

Uses both Pauli-even and Pauli-odd terms.

Operators and Observables

Action of the Dirac matrix operators become

$$\hat{\gamma}_\mu|\psi\rangle \quad \leftrightarrow \quad \gamma_\mu\psi\gamma_0 \quad (\mu = 0, \ldots, 3)$$
$$i|\psi\rangle \quad \leftrightarrow \quad \psi\, I\boldsymbol{\sigma}_3$$
$$\hat{\gamma}_5|\psi\rangle \quad \leftrightarrow \quad \psi\boldsymbol{\sigma}_3$$

Verification is routine computation.

Dirac theory replaces non-covariant Hermitian adjoint with Dirac adjoint

$$\langle\bar{\psi}| = (\langle\psi_u|, -\langle\psi_l|)$$

(u and l for upper and lower components.) The inner product decomposes to

$$\langle\bar{\psi}|\phi\rangle = \langle\psi_u|\phi_u\rangle - \langle\psi_l|\phi_l\rangle$$

This has the STA equivalent

$$\langle\psi_u^\dagger\phi_u\rangle_q - \langle\psi_l^\dagger\phi_l\rangle_q$$
$$= \langle(\tilde{\psi}_u - \boldsymbol{\sigma}_3\tilde{\psi}_l)(\psi_u + \psi_l\boldsymbol{\sigma}_3)\rangle_q = \langle\tilde{\psi}\phi\rangle_q$$

So Dirac adjoint $=$ spacetime reversion (covariant).

Now look at observables for a Dirac spinor. First is current

$$J_\mu = \langle\bar{\psi}|\hat{\gamma}_\mu|\psi\rangle \leftrightarrow \langle\tilde{\psi}\gamma_\mu\psi\gamma_0\rangle - \langle\tilde{\psi}\gamma_\mu\psi I\gamma_3\rangle I\boldsymbol{\sigma}_3.$$

But

$$(\psi I\gamma_3\tilde{\psi})^{\sim} = -\psi I\gamma_3\tilde{\psi} = \text{trivector}$$

and

$$(\psi\gamma_0\tilde{\psi})^{\sim} = \psi\gamma_0\tilde{\psi} = \text{vector}$$

So

$$J_\mu = \langle\bar{\psi}|\hat{\gamma}_\mu|\psi\rangle \;\leftrightarrow\; \gamma_\mu\!\cdot\!(\psi\gamma_0\tilde{\psi}).$$

Taking expectation value = picking out component of a vector. Reconstitute full vector $J = J_\mu\gamma^\mu$:

$$J = \psi\gamma_0\tilde{\psi}$$

Form Lorentz invariant decomposition of ψ. Write

$$\psi\tilde{\psi} = \text{scalar+pseudoscalar} = \rho\,\mathrm{e}^{I\beta}$$

Define spacetime rotor R by

$$R = \psi\rho^{-1/2}\,\mathrm{e}^{-I\beta/2}, \qquad R\tilde{R} = 1.$$

Have decomposed spinor ψ into

$$\psi = \rho^{1/2}\,\mathrm{e}^{I\beta/2}\,R$$

Has invariant density ρ, rotor R, and curious factor of β. Return to this for plane-wave states. Now have current

$$J = \psi\gamma_0\tilde{\psi} = \rho\,\mathrm{e}^{I\beta/2}\,R\gamma_0\tilde{R}\,\mathrm{e}^{I\beta/2} = \rho R\gamma_0\tilde{R}.$$

R boosts γ_0 onto direction of the current. Just like point particle model.

Similar picture for spin. Now a rank 2 antisymmetric tensor (a bivector!) in relativistic theory. Components given by observables

$$\langle\bar{\psi}|i\tfrac{1}{2}(\hat{\gamma}_\mu\hat{\gamma}_\nu - \hat{\gamma}_\nu\hat{\gamma}_\mu)|\psi\rangle \\ \leftrightarrow \langle\tilde{\psi}\gamma_\mu\wedge\gamma_\nu\psi I\boldsymbol{\sigma}_3\rangle_q = \langle\gamma_\mu\wedge\gamma_\nu\psi I\boldsymbol{\sigma}_3\tilde{\psi}\rangle$$

Imaginary component vanishes again. Picks out components of the spin bivector S,

$$S = \psi I\boldsymbol{\sigma}_3\tilde{\psi}$$

Natural generalisation of the Pauli result. Five such bilinear covariants in all

Grade	Standard	STA	Frame-Free
0	$\langle\bar{\psi}\|\psi\rangle$	$\langle\psi\tilde{\psi}\rangle$	$\rho\cos\beta$
1	$\langle\bar{\psi}\|\hat{\gamma}_\mu\|\psi\rangle$	$\gamma_\mu\cdot(\psi\gamma_0\tilde{\psi})$	$\psi\gamma_0\tilde{\psi} = J$
2	$\langle\bar{\psi}\|i\hat{\gamma}_{\mu\nu}\|\psi\rangle$	$(\gamma_\mu\wedge\gamma_\nu)\cdot(\psi I\boldsymbol{\sigma}_3\tilde{\psi})$	$\psi I\boldsymbol{\sigma}_3\tilde{\psi} = S$
3	$\langle\bar{\psi}\|\hat{\gamma}_\mu\hat{\gamma}_5\|\psi\rangle$	$\gamma_\mu\cdot(\psi\gamma_3\tilde{\psi})$	$\psi\gamma_3\tilde{\psi} = s$
4	$\langle\bar{\psi}\|i\hat{\gamma}_5\|\psi\rangle$	$\langle\psi\tilde{\psi}I\rangle$	$-\rho\sin\beta$

The Gyromagnetic Moment

Spin vector is $\rho R\gamma_3\tilde{R}$. Suggests a point-particle model.

Carry a frame with charged particle

$$e_\mu = R\gamma_\mu\tilde{R}, \quad e_0 = v, \quad e_3 = s$$

where s is (dimensionless) 'spin vector'. For rotor R use simplest form of Lorentz force law

$$\dot{R} = \frac{e}{2m}FR$$

Equations of motion for frame vectors are

$$\dot{e}_\mu = 2(\dot{R}\tilde{R})\cdot e_\mu = \frac{e}{m}F\cdot e_\mu$$

Now put particle at rest γ_0 frame, $v = \gamma_0$. Define

$$sv = s\wedge v = s\wedge\gamma_0 = \boldsymbol{s}$$

Equation for relative spin vector is

$$\dot{\boldsymbol{s}} = \frac{e}{m}(F\cdot s)\wedge\gamma_0 = \frac{e}{m}[F\cdot(\boldsymbol{s}\gamma_0)]\wedge\gamma_0$$

Now

$$F\cdot(\boldsymbol{s}\gamma_0) = \langle(\boldsymbol{E}+I\boldsymbol{B})\boldsymbol{s}\gamma_0\rangle_1 = \boldsymbol{E}\cdot\boldsymbol{s}\,\gamma_0 + (I\boldsymbol{B})\cdot\boldsymbol{s}\,\gamma_0,$$

so equation for $\boldsymbol{s}$ is

$$\dot{\boldsymbol{s}} = \frac{e}{m}(I\boldsymbol{B})\cdot\boldsymbol{s} = \frac{e}{m}\boldsymbol{s}\times\boldsymbol{B}$$

Precession equation for particle with gyromagnetic ratio of 2! The natural value for a relativistic frame in an electromagnetic field. Can also get Pauli version directly. Decompose R as

$$R = LU, \qquad L\gamma_0 = \gamma_0 \tilde{L}, \quad U\gamma_0 = \gamma_0 U$$

Rotor equation becomes

$$\begin{aligned} \dot{R}\tilde{R} = \dot{L}\tilde{L} + L\dot{U}\tilde{U}\tilde{L} &= \frac{e}{2m}F \\ \Longrightarrow \qquad \tilde{L}\dot{L} + \dot{U}\tilde{U} &= \frac{e}{2m}\tilde{L}FL \end{aligned}$$

U responds to de-boosted field $\tilde{L}FL$. Now set

$$L \approx 1 + \boldsymbol{v}/2, \qquad |\boldsymbol{v}| < 1$$

Work to $\mathrm{O}(\boldsymbol{v}^2)$, with $F = I\boldsymbol{B}$ a magnetic field in the laboratory frame (γ_0). Get

$$\tilde{L}FL = I\boldsymbol{B} - \boldsymbol{v}\cdot(I\boldsymbol{B}) + \mathrm{O}(\boldsymbol{v}^2)$$

Only relative bivector is $I\boldsymbol{B}$. Also relative bivector part of $\tilde{L}\dot{L} = \mathrm{O}(\boldsymbol{v}^2)$ so get equation

$$\dot{U}\tilde{U} = \frac{e}{2m}I\boldsymbol{B}, \quad \Longrightarrow \quad \dot{U} = \frac{e}{2m}I\boldsymbol{B}U$$

Replace proper-time τ with t, get non-relativistic rotor version of Pauli equation. Compare directly and again see $g = 2$ as

$$\gamma = g\frac{e}{2m} = \frac{e}{m}$$

The Dirac Equation

Quantum mechanics deals with wave equations. Need a relativistic wave equation for Dirac spinor ψ. Again, has single-sided rotor transformation law, $\psi \mapsto R\psi$. Must put covariant vectors on left, Can put γ_0 γ_3 $I\boldsymbol{\sigma}_3$ on right. First guess

$$\nabla\psi = 0$$

STA generalisation of Cauchy-Riemann equations. The neutrino wave equation. Solution decompose into

$$\psi = \psi\tfrac{1}{2}(1+\boldsymbol{\sigma}_3) + \psi\tfrac{1}{2}(1-\boldsymbol{\sigma}_3) = \psi_+ + \psi_-.$$

ψ_+ and ψ_- are right and left-handed helicity eigenstates. (Nature only appears to use left-handed — need electroweak theory.)

Operator identification $i\partial_\mu = p_\mu$, so wavefunction for a free massive particle should have $\nabla^2\psi = -m^2\psi$. Put term on right, linear in m. For plane-wave states, momentum p, get

$$p\psi = m\psi a_0$$

where a_0 to be determined. Must be odd grade and square to $+1$. Obvious choice: $a_0 = \gamma_0$. Gives

$$\nabla\psi I\boldsymbol{\sigma}_3 = m\psi\gamma_0$$

or

$$\nabla\psi = -m\psi I\gamma_3.$$

The Dirac equation in STA form. Convert back to matrix form with

$$\nabla\psi\gamma_0 \leftrightarrow \hat{\gamma}^\mu\partial_\mu|\psi\rangle$$

Now have a first-order wave equation, with observables formed from ψ. Most important is current $J = \psi\gamma_0\tilde{\psi}$. Satisfies

$$\begin{aligned}\nabla\cdot J &= \langle\nabla\psi\gamma_0\tilde{\psi}\rangle + \langle\psi\gamma_0\dot{\tilde{\psi}}\dot{\nabla}\rangle \\ &= -m\langle\psi I\boldsymbol{\sigma}_3\tilde{\psi} + \psi(\psi I\boldsymbol{\sigma}_3)^{\sim}\rangle = 0\end{aligned}$$

so is conserved. Single fermions cannot be created or destroyed. Fermion pairs, *eg* electron + positron can annihilate, but that is quantum field theory. The timelike component of J is

$$J_0 = \gamma_0\cdot J = \langle\gamma_0\tilde{\psi}\gamma_0\psi\rangle = \langle\psi^\dagger\psi\rangle > 0$$

so positive definite. Interpreted as a probability density, so have a relativistic theory with a consistent probabilistic interpretation. This was Dirac's original goal.

Plane-Wave States

Positive energy plane-wave state defined by

$$\psi = \psi_0 \, \mathrm{e}^{-I\boldsymbol{\sigma}_3 p\cdot x}$$

Dirac equation gives

$$p\psi_0 = m\psi_0\gamma_0$$

so

$$p\psi_0\tilde{\psi}_0 = mJ.$$

But can write

$$\psi\tilde{\psi} = \rho \, \mathrm{e}^{i\beta} = \psi_0\tilde{\psi}_0$$

So $\exp(i\beta) = \pm 1$. Since positive energy need $\beta = 0$. ψ_0 is rotor + normalisation constant. Proper boost L from $m\gamma_0$ to p/m:

$$p = mL\gamma_0\tilde{L} = mL^2\gamma_0,$$

Solved by

$$L = \frac{m + p\gamma_0}{[2m(m + p\cdot\gamma_0)]^{1/2}} = \frac{E + m + \boldsymbol{p}}{[2m(E+m)]^{1/2}},$$

where $p\gamma_0 = E + \boldsymbol{p}$. Full spinor ψ_0 is LU, with U is a spatial (Pauli) rotor.

Negative energy have phase factor $\mathrm{e}^{+I\boldsymbol{\sigma}_3 p\cdot x}$, with $E = \gamma_0 \cdot p > 0$. For these $-p\psi\tilde{\psi} = mJ$ so need $\beta = \pi$.

Summarise by

positive energy $\quad \psi^{(+)}(x) = L(p)U\,\mathrm{e}^{-I\boldsymbol{\sigma}_3 p\cdot x}$

negative energy $\quad \psi^{(-)}(x) = L(p)UI\,\mathrm{e}^{I\boldsymbol{\sigma}_3 p\cdot x}$

Crucial in scattering theory. NB 'curious' β factor is particle/antiparticle mixing ratio. (More complicated for bound states).

Hamiltonian Form

Borrow 'i' symbol as abbreviation for right-sided multiplication by $I\boldsymbol{\sigma}_3$. Pre-multiply Dirac equation by γ_0, $\gamma_0\nabla = \partial_t + \boldsymbol{\nabla}$,

$$i\partial_t\psi = -\boldsymbol{\nabla}\psi I\boldsymbol{\sigma}_3 + m\gamma_0\psi\gamma_0 = -i\boldsymbol{\nabla}\psi + m\bar{\psi}.$$

where $\bar{\psi} = \gamma_0\psi\gamma_0$. Right-hand side defines Hamiltonian, $\mathcal{H}$,

$$\mathcal{H}\psi = -i\boldsymbol{\nabla}\psi + \bar{\psi}.$$

γ_0 in definition shows that the Hamiltonian is observer dependent. Classify states in terms of eigenstates of operators which commute with the Hamiltonian. Accompanying quantum numbers conserved in time.

Angular Operators

Angular momentum operators $\hat{L}_i$, defined by

$$\hat{L}_i = -i\epsilon_{ijk} x_j \partial_k.$$

Components of bivector operator $i\boldsymbol{x}\wedge\boldsymbol{\nabla}$. Form STA equivalent

$$L_B = iB\cdot(\boldsymbol{x}\wedge\boldsymbol{\nabla}),$$

B a relative bivector. $B = I\boldsymbol{\sigma}_i$ recovers component form. The L_B satisfy commutation relation (exercise)

$$[L_{B_1}, L_{B_2}] = -iL_{B_1\times B_2}.$$

Directly encodes the bivector commutation relations, so relates back to the rotation group — a symmetry group.

Form commutator of L_B with Hamiltonian $\mathcal{H}$. Commutes with the bar operator $\psi \mapsto \bar{\psi}$, but

$$[L_B, \mathcal{H}] = [B\cdot(\boldsymbol{x}\wedge\boldsymbol{\nabla}), \boldsymbol{\nabla}] = -\dot{\boldsymbol{\nabla}} B\cdot(\dot{\boldsymbol{x}}\wedge\boldsymbol{\nabla}) = B\cdot\boldsymbol{\nabla}$$

Orbital angular momentum not conserved in relativistic physics. But

$$B\cdot\boldsymbol{\nabla} = \tfrac{1}{2}(B\boldsymbol{\nabla} - \boldsymbol{\nabla}B)$$

So

$$[B\cdot(\boldsymbol{x}\wedge\boldsymbol{\nabla}) - \tfrac{1}{2}B, \mathcal{H}] = 0.$$

Get conserved angular momentum operator

$$J_B = L_B - \tfrac{1}{2}iB.$$

In conventional notation:

$$\hat{J}_i = \hat{L}_i + \tfrac{1}{2}\hat{\Sigma}_i$$

where $\hat{\Sigma}_i = (i/2)\epsilon_{ijk}\hat{\gamma}_j\hat{\gamma}_k$. Extra term $\frac{1}{2}B$ defines "spin-1/2". Look for eigenstates of J_3 operator. Spin contribution is

$$-\tfrac{1}{2}iI\boldsymbol{\sigma}_3\psi = \tfrac{1}{2}\boldsymbol{\sigma}_3\psi\boldsymbol{\sigma}_3.$$

In Pauli theory eigenstates are 1 and $-I\boldsymbol{\sigma}_2$, eigenvalues $\pm\frac{1}{2}$.

In the relativistic theory separate spin and orbital operators not conserved. Only combined J_B that commute with $\mathcal{H}$.

- Result rests solely on commutation properties of $B\cdot(\boldsymbol{x}\wedge\boldsymbol{\nabla})$ and $\boldsymbol{\nabla}$. Factor of $1/2$ has no special relation to 3-d rotation group.
- $J_B = L_B - \frac{1}{2}iB$ is scalar + bivector. Standard notation encourages view of this as sum of two vector operators!