February 25, 1999

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 13

SUMMARY

The standard model of particle physics describes interactions in terms of gauge theories. But one force is not included — gravity. It has been known for some time that gravity can be formulated as a gauge theory, but thought to be difficult and unnatural. Geometric algebra changes this view. Theory becomes much easier — and no longer have to worry about curved space.

1. Electromagnetism as a gauge theory.

- Local phase transformations, and gauge fields.
- Covariant derivatives and minimal coupling.

2. Gravity as a gauge theory in GA.

- Principles. All interactions described by fields. Properties of the underlying STA are unmeasurable.
- Gauging displacements.
- Gauging rotations.
- The Dirac equation coupled to gravity.

Central Potentials

Add term $V(r)$ to the Hamiltonian. The J_B operators still commute with $\mathcal{H}$, since $\boldsymbol{x}\wedge\boldsymbol{\nabla} V(r) = 0$. Key objects are analytic functions in 3-d — Pauli spinors satisfying

$$\boldsymbol{\nabla}\Psi = 0.$$

Have $\boldsymbol{\nabla}^2\Psi = 0$ so components are spherical harmonics. Radial dependence goes as r^l, l an integer. Separate out radial and angular dependence:

$$\Psi = r^l \psi(\theta, \phi)$$

(r, θ, ϕ) are spherical polar coordinates. See that

$$0 = \boldsymbol{x}\boldsymbol{\nabla}\Psi = r\partial_r\Psi + \boldsymbol{x}\wedge\boldsymbol{\nabla}\Psi,$$

so ψ satisfies eigenvalue equation

$$-\boldsymbol{x}\wedge\boldsymbol{\nabla}\psi = l\psi.$$

$\psi(\theta, \phi)$ can be simultaneous eigenstates of $\boldsymbol{x}\wedge\boldsymbol{\nabla}$ and one of J_B, since commute. Choose J_3 and write

$$-\boldsymbol{x}\wedge\boldsymbol{\nabla}\psi_l^m = l\psi_l^m \qquad l \geq 0$$

$$J_3\psi_l^m = (m + \tfrac{1}{2})\psi_l^m \qquad -1 - l \leq m \leq l.$$

Eigenstates with negative l constructed from $\boldsymbol{\sigma}_r = \boldsymbol{x}/r$,

$$-\boldsymbol{x}\wedge\boldsymbol{\nabla}(\boldsymbol{\sigma}_r\psi_l^m\boldsymbol{\sigma}_3) = -(l+2)\boldsymbol{\sigma}_r\psi_l^m\boldsymbol{\sigma}_3.$$

Define non-zero integers $\kappa = l+1$, so that κ is a non-zero integer. Degeneracy is $|2\kappa|$.

Construct eigenfunctions of the Dirac Hamiltonian

$$E\psi = \mathcal{H}\psi = -\boldsymbol{\nabla}\psi I\boldsymbol{\sigma}_3 + eV(r)\psi + m\gamma_0\psi\gamma_0$$

as

$$\psi(\boldsymbol{x},\kappa) = \psi_l^m u(r) + \boldsymbol{\sigma}_r\psi_l^m v(r) I\boldsymbol{\sigma}_3$$

where u and $v(r)$ are complex superpositions of $1, I\boldsymbol{\sigma}_3$. Radial equations separate to give

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} (\kappa-1)/r & -(E-eV(r)+m) \\ E-eV(r)-m & (-\kappa-1)/r \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

The Hydrogen Atom

Set $eV = -Z\alpha/r$, where $\alpha = e^2/4\pi$ is the fine structure constant, $\approx 1/137$, and Z is atomic charge. Find energy spectrum

$$E^2 = m^2\left[1 - \frac{(Z\alpha)^2}{n'^2 + 2n'\nu + (l+1)^2}\right]$$

where n' is non-negative integer, m is electron mass, and

$$\nu = [(l+1)^2 - (Z\alpha)^2]^{1/2}.$$

Get non-relativistic formula from

$$\nu \approx l + 1$$

and

$$E \approx m\left[1 - \frac{(Z\alpha)^2}{2}\frac{1}{n'^2 + 2n'(l+1) + (l+1)^2}\right]$$

Subtract off rest-mass energy. Get non-relativistic expression

$$E_{NR} = -m\frac{(Z\alpha)^2}{2}\frac{1}{(n'+l+1)^2} = -\frac{mZ^2e^4}{32\pi^2\epsilon_0^2\hbar^2}\frac{1}{n^2}$$

where $n = n' + l + 1$. Get familiar Bohr formula for energy levels.

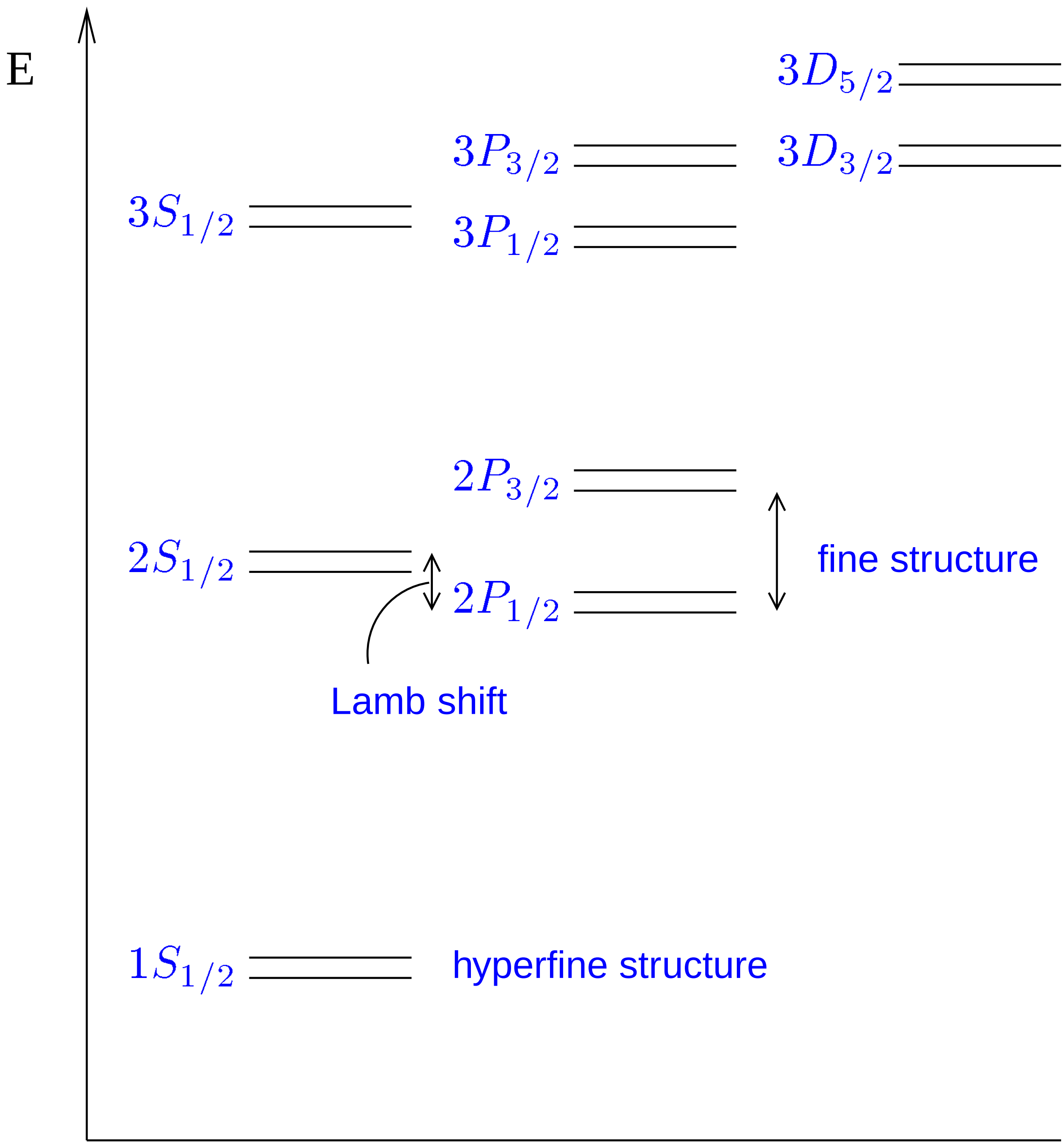
Expanding to next order get

$$E_{NR} = -m\frac{(Z\alpha)^2}{2n^2} - m\frac{(Z\alpha)^4}{2n^4}\left(\frac{n}{l+1} - \frac{3}{4}\right).$$

Binding energy increased slightly, and get dependence on l.

Lifts degeneracy in non-relativistic solution.

H-Atom Energy Levels



Dirac equation accounts for fine structure. The hyperfine structureis due to interaction with the magnetic moment of the nucleus. The Lamb shift is explained by quantum field theory.

ELECTROMAGNETISM AS A GAUGE THEORY

Start with

$$\nabla\psi I\boldsymbol{\sigma}_3 = m\psi\gamma_0 \tag{1}$$

A global symmetry of this is

$$\psi \mapsto \psi' = \psi\, \mathrm{e}^{I\boldsymbol{\sigma}_3\theta} \tag{2}$$

where $\theta = \text{constant}$. Clearly ψ' is a solution of (1) if ψ is.

But what if $\theta = \theta(x)$? Then, writing $R = \mathrm{e}^{I\boldsymbol{\sigma}_3\theta}$, have

$$\nabla\psi' = (\nabla\psi)R + (\nabla\theta)\psi R I\boldsymbol{\sigma}_3.$$

and so

$$\nabla\psi' I\boldsymbol{\sigma}_3 \neq m\psi'\gamma_0.$$

This means the symmetry (2) does not work locally.

Why should we want it to?

— from the structure of the physical statements of the Dirac theory.

These are of two main types:

(i) The values of observables. Formed via inner products

$$\langle\psi|\phi\rangle \;\leftrightarrow\; \langle\tilde{\psi}\phi\rangle - \langle\tilde{\psi}\phi I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3$$

(ii) statements of equality like $\psi = \psi_1 + \psi_2$.

The physical content of both these equations is unchanged if all the spinors are rotated by the same locally varying phase factor.

Our theory should be invariant under such changes.

Covariant Derivatives

To achieve this, have to change the ∇ operator to get rid of unwanted term in gradient of R.

Putting $\nabla = \partial_a a\cdot\nabla = \gamma^\mu \partial_\mu$, equation for $\nabla\psi'$ is

$$\nabla\psi' = \partial_a \left(a\cdot\nabla\psi R + \psi a\cdot\nabla R\right).$$

It is the last term which does the damage.

Therefore define a new operator D via

$$D\psi = \partial_a \left(a\cdot\nabla\psi + \tfrac{1}{2}\psi\Omega(a)\right)$$

and a new Dirac equation

$$D\psi I\boldsymbol{\sigma}_3 = m\psi\gamma_0$$

and see what properties $\Omega(a)$ must have to remove unwanted term.

Under $\psi \mapsto \psi' = \psi R$ will have $D \mapsto D'$ where D' should have the same form as D. For general ϕ, set

$$D'\phi = \partial_a \left(a\cdot\nabla\phi + \tfrac{1}{2}\phi\Omega'(a)\right)$$

Our basic requirement is that $\psi' = \psi R$ should solve the Dirac equation with D' instead of D, if ψ solves the equation with D.

This will work if

$$D'\psi' = D'(\psi R) = (D\psi)R, \qquad (3)$$

since then

$$\begin{aligned} D'\psi' I\boldsymbol{\sigma}_3 - m\psi'\gamma_0 &= (D\psi)RI\boldsymbol{\sigma}_3 - m\psi R\gamma_0 \\ &= (-m\psi i\gamma_3)RI\boldsymbol{\sigma}_3 - m\psi R\gamma_0 = 0 \end{aligned}$$

Can see generally that (3) is the right thing

— we want a D that suppresses the differentiation of R.

So let's try our forms for D and D'. We get

$$\begin{aligned} D'(\psi R) &= \partial_a \left(a\cdot\nabla\psi R + \psi a\cdot\nabla R + \tfrac{1}{2}\psi R\Omega'(a)\right) \\ &= D\psi R = \partial_a \left(a\cdot\nabla\psi + \frac{1}{2}\psi\Omega(a)\right) R. \end{aligned}$$

Identifying terms, we must have

$$a\cdot\nabla R + \frac{1}{2}R\Omega'(a) = \frac{1}{2}\Omega(a)R,$$

i.e.

$$\Omega'(a) = \tilde{R}\Omega(a)R - 2\tilde{R}a\cdot\nabla R$$

This gives the transformation property of $\Omega(a)$

— what type of object is it?

$$\tilde{R}R = 1 \implies a\cdot\nabla\tilde{R}R + \tilde{R}a\cdot\nabla R = 0$$

i.e. $\tilde{R}a\cdot\nabla R = -\left(\tilde{R}a\cdot\nabla R\right)^{\sim}$ which is therefore a bivector.

Thus $\Omega(a)$ must be a bivector field. It is called a connection and belongs to the Lie algebra of the symmetry group.

In general $\Omega(a)$ will not be expressible as the derivative of a rotor field. This is the essence of the gauging step. Take something arising from a derivative, and generalize it to a term that cannot be formed this way.

MINIMALLY-COUPLED DIRAC EQUATION

Now restrict rotation to $\gamma_2\gamma_1$ plane using $R = \mathrm{e}^{I\boldsymbol{\sigma}_3\theta}$. Then

$$\begin{aligned}-2\tilde{R}a\cdot\nabla R &= -2\,\mathrm{e}^{-I\boldsymbol{\sigma}_3\theta}\,a\cdot(\nabla\theta)\,\mathrm{e}^{I\boldsymbol{\sigma}_3\theta}\,I\boldsymbol{\sigma}_3 \\ &= -2a\cdot(\nabla\theta)I\boldsymbol{\sigma}_3.\end{aligned}$$

Generalizing this, we can deduce $\Omega(a) = \lambda a\cdot A I\boldsymbol{\sigma}_3$ where A is a general 4-vector and λ a coupling constant.

Note if A were equal to $\nabla\theta$, then $\nabla\wedge A = 0$. Will generalise this when we look at the field strength tensor. Now have

$$D\psi = \partial_a(a\cdot\nabla\psi + \tfrac{1}{2}\lambda a\cdot A\, I\boldsymbol{\sigma}_3) = \nabla\psi + \tfrac{1}{2}\lambda A\psi I\boldsymbol{\sigma}_3.$$

Introduces term $\lambda\gamma_0 A/2$ in Hamiltonian. Scalar part $\lambda V/2$, so for electron require $\lambda = 2e$.

Get 'minimally coupled' Dirac equation

$$\nabla \psi I \boldsymbol{\sigma}_3 - eA\psi = m\psi\gamma_0.$$

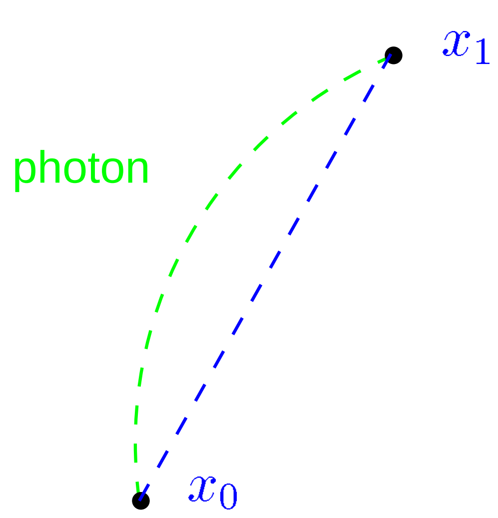
This is simplest (minimal) possible modification to original equation. No extra terms in $F\psi$ or $F^2\psi$ (all acceptable). Nature appears to be 'minimal' in its principles.

Gauge Principles for Gravitation

Aim: To model gravitational interactions in terms of (gauge) fields defined in the STA.

A radical departure from GR! The STA is the geometric algebra of flat spacetime. Extra fields cannot change this.

But what about standard arguments that spacetime is curved? These all involve light paths, or measuring rods. All modeled with interacting fields. So photon paths need not be 'straight'.



The STA vector $x_1 - x_0$ has no measurable significance now.

This will follow if we ensure that all physical predictions are independent of the absolute position and orientation of fields in the STA. Only relations between fields are important.

Becomes clearer if we consider fields. Take spinors $\psi_1(x)$ and $\psi_2(x)$. A sample physical statement is

$$\psi_1(x) = \psi_2(x).$$

At a point where one field has a particular value, the second field has the same value. This is independent of where we place the fields in the STA. And independent of where we choose to locate other values of the fields. Could equally well introduce two new fields

$$\psi_1'(x) = \psi_1(x'), \quad \psi_2'(x) = \psi_2(x'),$$

with x' an arbitrary function of x. Equation $\psi_1'(x) = \psi_2'(x)$ has precisely the same physical content as original.

Same is true if act on fields with a spacetime rotor

$$\psi_1' = R\psi_1, \quad \psi_2' = R\psi_2$$

Again, $\psi_1' = \psi_2'$ has same physical content as original equation. Same true of observables, eg $J = \psi\gamma_0\tilde{\psi}$. Now $\psi \mapsto \psi'$ produces the new vector $J' = RJ\tilde{R}$. Hence absolute direction irrelevant.

DISPLACEMENTS

We write

$$x' = f(x)$$

for an arbitrary (differentiable) map between spacetime position vectors. A rule for relating position vectors in same space — not a map between manifolds.

Use this to move field $\psi(x)$ to new field

$$\psi'(x) \equiv \psi(x').$$

Call this a displacement. A better name than '*translation*' (too rigid) or '*diffeomorphism*' (too technical).

Now consider behaviour of derivative of ψ, $\nabla\psi = \partial_a \, a\cdot\nabla\psi$. See that

$$\begin{aligned} a\cdot\nabla\psi'(x) &= a\cdot\nabla\psi[f(x)] \\ &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(\psi[f(x+\epsilon a)] - \psi[f(x)]\right) \\ &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(\psi[f(x)+\epsilon \mathsf{f}(a)] - \psi[f(x)]\right). \end{aligned}$$

where

$$\mathsf{f}(a) = \mathsf{f}(a;x) = a\cdot\nabla f(x)$$

and have Taylor expanded $f(x+\epsilon a)$ to first order. $\mathsf{f}(a)$ is linear on a. Suppress position dependence where possible.

Now have

$$a\cdot\nabla\psi'(x) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\psi[x'+\epsilon \mathsf{f}(a)] - \psi(x')\right).$$

But this is the vector derivative with respect to x' in $\mathsf{f}(a)$ direction

$$a\cdot\nabla\psi'(x) = \mathsf{f}(a)\cdot\nabla_{x'}\psi(x'),$$

where $\nabla_{x'}$ is derivative with respect to the new vector position variable x'. Since

$$\mathsf{f}(a)\cdot\nabla_{x'} = a\cdot\bar{\mathsf{f}}(\nabla_{x'})$$

get operator relation

$$\nabla_x = \bar{\mathsf{f}}(\nabla_{x'})$$

$\mathsf{f}(a)$ is coordinate-free form of Jacobian.

Now suppose we have a physical relation such as

$$\nabla\phi = A.$$

Scalar ϕ and vector A. (Eg A is pure electromagnetic gauge). Now replace $\phi(x)$ by $\phi'(x) = \phi(x')$ and $A(x)$ by $A'(x) = A(x')$. Left-hand side becomes

$$\nabla\phi'(x) = \bar{\mathsf{f}}(\nabla_{x'})\phi(x') = \bar{\mathsf{f}}[A(x')] = \bar{\mathsf{f}}(A')$$

so no longer equal to A'.

Gauge field must assemble with ∇ to form object which, under displacements, re-evaluates to derivative with respect to the new position vector. Replace ∇ with $\bar{\mathsf{h}}(\nabla)$, with

$$\bar{\mathsf{h}}(a) = \bar{\mathsf{h}}(a; x)$$

an arbitrary function of position, and a linear function of a. Property we require is that

$$\bar{\mathsf{h}}'(\nabla\phi') = \bar{\mathsf{h}}'[\bar{\mathsf{f}}(A')] = \bar{\mathsf{h}}(A'; x')$$

Suppressing position dependence, basic requirement is

$$\bar{\mathsf{h}}'(a) = \bar{\mathsf{h}}\bar{\mathsf{f}}^{-1}(a)$$

for general vector a. Now systematically replace ∇ by $\bar{\mathsf{h}}(\nabla)$. Get all equations invariant under displacements. Eg. Dirac equation is now

$$\bar{\mathsf{h}}(\nabla)\psi I\boldsymbol{\sigma}_3 = m\psi\gamma_0.$$

$\bar{\mathsf{h}}$-field not a connection in conventional Yang-Mills sense. But $\bar{\mathsf{h}}$-field does ensure that a symmetry is local, so still called a gauge field.

Rotations

Second symmetry we require is invariance under

$$\psi \mapsto \psi' = R\psi, \qquad (4)$$

where R is an arbitrary, position-dependent rotor in spacetime. (Say that R as generates rotations. Understood that this includes boosts.) Back in familiar territory now! Write

$$\bar{\mathsf{h}}(\nabla)\psi = \bar{\mathsf{h}}(\partial_a)\, a\cdot\nabla\psi.$$

To make (4) a symmetry, modify $a\cdot\nabla$ by adding a bivector connection $\Omega(a)$,

$$D_a\psi = a\cdot\nabla + \tfrac{1}{2}\Omega(a)\psi$$

where $\Omega(a)$ has the transformation law

$$\Omega(a) \mapsto \Omega'(a) = R\Omega(a)\tilde{R} - 2a\cdot\nabla R\tilde{R}.$$

Since R is an arbitrary rotor, now no constraint on the terms in $\Omega(a)$. Has $\Omega(a)$ has $6 \times 4 = 24$ degrees of freedom.

Equation now reads

$$D\psi I\boldsymbol{\sigma}_3 = \bar{\mathsf{h}}(\partial_a)D_a\psi I\boldsymbol{\sigma}_3 = m\psi\gamma_0. \qquad (5)$$

Replace ψ by ψ' and $\Omega(a)$ by $\Omega'(a)$, find that the left-hand side becomes

$$\bar{\mathsf{h}}(\partial_a)D'_a(R\psi)I\boldsymbol{\sigma}_3 = \bar{\mathsf{h}}(\partial_a)RD_a\psi I\boldsymbol{\sigma}_3$$

But right-hand side is simply $mR\psi\gamma_0$. Need to transform the $\bar{\mathsf{h}}$-field as well,

$$\bar{\mathsf{h}}(a) \mapsto \bar{\mathsf{h}}'(a) = R\bar{\mathsf{h}}(a)\tilde{R}.$$

This is sensible. Recall $\bar{\mathsf{h}}(\nabla\phi) = A$. Invariant under displacements. Also invariant if both vectors are rotated. But rotation of $\bar{\mathsf{h}}(\nabla\phi)$ must be driven by transforming $\bar{\mathsf{h}}$.

Dirac equation now invariant under both rotations and displacements. Achieved by introducing two new gauge fields, $\bar{\mathsf{h}}(a)$ and $\Omega(a)$. A total of $16 + 24 = 40$ degrees of freedom!