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PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 14

SUMMARY

The key to deriving the field equations in a gauge theory is the covariant field strength tensor. In tis lecture we study the properties of these for the two gravitational gauge fields.

- Commutators of covariant derivatives and the field strength.
- The gravitational field strength and non-linearity.
- The Riemann tensor, and some simple examples.
- The Planck scale and the magnitude of the field strength of the displacement field.
- Further properties of the Riemann tensor.
- The second field equation.

The Field Strength

Form commutator of covariant derivatives. First take electromagnetism, $\psi \mapsto \psi R$. With a and b constant vectors, get

$$[D_a, D_b]\psi = \frac{1}{2}\psi[a\cdot\nabla\Omega(b) - b\cdot\nabla\Omega(a) - \Omega(a)\times\Omega(b)]$$

All derivatives of ψ have canceled.

Restricting to $\Omega(a) = -2a\cdot A\, I\boldsymbol{\sigma}_3$, get term in

$$b\cdot\nabla(a\cdot A\, I\boldsymbol{\sigma}_3) - a\cdot\nabla(b\cdot A\, I\boldsymbol{\sigma}_3) - 2a\cdot A\, b\cdot A\, I\boldsymbol{\sigma}_3\times I\boldsymbol{\sigma}_3$$
$$= (a\wedge b)\cdot(\nabla\wedge A)I\boldsymbol{\sigma}_3 = (a\wedge b)\cdot F\, I\boldsymbol{\sigma}_3$$

Maps bivector $a\wedge b$ linearly onto a pure phase term. In electromagnetism lose mapping and extract $F = \nabla\wedge A$. This is physical field. Vanishes if A is pure gauge.

Rotation Gauge

For rotations, rotors multiply ψ from left, so

$$[D_a, D_b]\psi = \frac{1}{2}\mathsf{R}(a\wedge b)\psi$$

where

$$\mathsf{R}(a\wedge b) = a\cdot\nabla\Omega(b) - b\cdot\nabla\Omega(a) + \Omega(a)\times\Omega(b)$$

Right-hand side is antisymmetric on a, b, so a linear funntion of the bivector $a\wedge b$. Extend to general bivectors

$$\mathsf{R}(a\wedge b + c\wedge d) = \mathsf{R}(a\wedge b) + \mathsf{R}(c\wedge d).$$

Can write the field strength as,

$$\mathsf{R}(B) = \mathsf{R}(B;x)$$

- A position dependent, linear function of the bivector B. Returns a general bivector, so $6\times 6 = 36$ Degrees of freedom.
- Term in $\Omega(a)\times\Omega(b)$ is non-linear. Cannot superpose two solutions to get a third. Much more difficult than electromagnetism.

Transformation properties easy to establish

$$\begin{aligned}[D'_a, D'_b,]\psi' &= \tfrac{1}{2}\mathsf{R}'(a\wedge b)R\psi \\ &= R[D_a, D_b]\psi = \tfrac{1}{2}R\mathsf{R}(a\wedge b)\psi\end{aligned}$$

so read off that

$$\mathsf{R}'(a\wedge b) = R\mathsf{R}(a\wedge b)\tilde{R}.$$

Field strength now transforms under gauge transformations.

If $\Omega(a)$ pure gauge, then can find gauge where $D_a = a\cdot\nabla$. $\mathsf{R}(a\wedge b)$ vanishes in this gauge, so vanishes in all gauges.

Displacement Gauge

$\bar{\mathsf{h}}(a)$ couples to derivatives differently. Form commutator, now get

$$[a\cdot\bar{\mathsf{h}}(\nabla), b\cdot\bar{\mathsf{h}}(\nabla)]\psi = (b\wedge a)\cdot[\bar{\mathsf{h}}(\nabla)\wedge\bar{\mathsf{h}}(\nabla)]\psi.$$

Now do get a differential operator. driven by $\bar{\mathsf{h}}(\nabla)\wedge\bar{\mathsf{h}}(\nabla)$ term. Acting on scalar ϕ

$$\begin{aligned}\bar{\mathsf{h}}(\nabla)\wedge\bar{\mathsf{h}}(\nabla\phi) &= \bar{\mathsf{h}}(\dot{\nabla})\wedge\dot{\bar{\mathsf{h}}}(\nabla\phi) + \bar{\mathsf{h}}(\nabla\wedge\nabla\phi) \\ &= \bar{\mathsf{h}}(\dot{\nabla})\wedge\dot{\bar{\mathsf{h}}}(\nabla\phi)\end{aligned}$$

where over-dot again denotes scope of derivative. Now $\bar{\mathsf{h}}(\nabla\phi)$ is covariant. So generalise to

$$\mathsf{S}(a) = \bar{\mathsf{h}}(\dot{\nabla})\wedge\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(a) = -\bar{\mathsf{h}}[\nabla\wedge\bar{\mathsf{h}}^{-1}(a)],$$

where a is a constant vector, and have used

$$\nabla\wedge[\bar{\mathsf{h}}\bar{\mathsf{h}}^{-1}(a)] = \dot{\nabla}\wedge\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(a) + \dot{\nabla}\wedge\bar{\mathsf{h}}\dot{\bar{\mathsf{h}}}^{-1}(a) = \nabla\wedge a = 0.$$

$\mathsf{S}(a)$ is covariant under displacements. What about pure gauge? In this case

$$\bar{\mathsf{h}}(a) = \bar{\mathsf{f}}^{-1}(a),$$

so that $\bar{\mathsf{h}}(\nabla) = \bar{\mathsf{f}}^{-1}(\nabla) = \nabla_{x'}$ is vector derivative in some other gauge. For this case

$$\mathsf{S}(a) = -\bar{\mathsf{f}}^{-1}[\nabla\wedge\bar{\mathsf{f}}(a)].$$

But $\mathsf{f}(a)$ is the derivative of a displacement,

$$\mathsf{f}(a) = a\cdot\nabla f(x),$$

so

$$\bar{\mathsf{f}}(a) = \partial_b\langle\mathsf{f}(b)a\rangle = \partial_b\langle b\cdot\nabla f(x)a\rangle = \nabla(a\cdot f(x)).$$

Henced $\nabla\wedge\bar{\mathsf{f}}(a) = \nabla\wedge\nabla(a\cdot f(x)) = 0$. $\mathsf{S}(a)$ has the desired properties.

Covariant Field Strengths

Need covariant forms of field strength. Start with rotation gauge. $\Omega(a)$ removes terms in $a\cdot\nabla R\tilde{R}$. Under displacements must have

$$\Omega(a;x) \mapsto \Omega'(a;x) = \Omega[\mathsf{f}(a);x']$$

Field strength has term in $\Omega(a)\times\Omega(b)$. Must transform to

$$\mathsf{R}(a\wedge b) \mapsto \mathsf{R}'(a\wedge b) = \mathsf{R}[\mathsf{f}(a)\wedge\mathsf{f}(b)] = \mathsf{R}[\mathsf{f}(a\wedge b);x']$$

Picks up a term in $\mathsf{f}(B)$. Remove this with suitable version of $\bar{\mathsf{h}}(a)$. Has

$$\bar{\mathsf{h}}(a) \mapsto \bar{\mathsf{h}}'(a) = \bar{\mathsf{h}}\bar{\mathsf{f}}^{-1}(a)$$

so adjoint transforms as

$$\mathsf{h}(a) \mapsto \mathsf{h}'(a) = \mathsf{f}^{-1}\mathsf{h}(a).$$

Insert this into $\mathsf{R}(B)$. Define covariant field strength

$$\mathcal{R}(B) = \mathsf{R}[\mathsf{h}(B)]$$

Factor of $\mathsf{h}(B)$ alters rotation gauge properties.

$$\bar{\mathsf{h}}(a) \mapsto \bar{\mathsf{h}}'(a) = R\bar{\mathsf{h}}(a)\tilde{R}.$$

so adjoint goes as

$$\mathsf{h}(a) \mapsto \mathsf{h}'(a) = \partial_b \langle aR\bar{\mathsf{h}}(b)\tilde{R}\rangle = \mathsf{h}(\tilde{R}aR).$$

Summarised transformation properties of $\mathcal{R}(B)$ by:

Displacements:	$\mathcal{R}'(B,x) = \mathcal{R}(B,x')$
Rotations:	$\mathcal{R}'(B) = R\mathcal{R}(\tilde{R}BR)\tilde{R}.$

Just what we want for a covariant tensor. Call $\mathcal{R}(B)$ the Riemann tensor. Understand rotation transformation from

$$\mathcal{R}(B) = \alpha B$$

This is ‘dilate all fields by factor α’. Transformed field is

$$\mathcal{R}'(B) = R\mathcal{R}(\tilde{R}BR)\tilde{R} = R(\alpha\tilde{R}BR)\tilde{R} = \alpha B$$

Same physical information.

EXAMPLES

I. The Schwarzschild Solution

Spherically symmetric source, mass M at rest in γ_0 frame, has

$$\mathcal{R}(B) = -\frac{M}{2r^3}(B + 3\boldsymbol{\sigma}_r B \boldsymbol{\sigma}_r)$$

where $r = |x \wedge \gamma_0|$, $\boldsymbol{\sigma}_r = x \wedge \gamma_0 / r$'. $M/2r^3$ controls the tidal force. In empty space these are measurable.

II. The Kerr Solution

Outside a rotating black hole get

$$\mathcal{R}(B) = -\frac{M}{2(r + IL\cos\theta)^3}(B + 3\boldsymbol{\sigma}_r B \boldsymbol{\sigma}_r).$$

Get Schwarzschild by $r \mapsto r + IL\cos\theta$. Explains complex structure in Kerr solution!

III. Cosmic Strings

Infinite, pressure-free string along γ_3, density ρ has

$$\mathcal{R}(B) = 8\pi\rho\langle BI\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3$$

Get tidal forces in $I\boldsymbol{\sigma}_3$ plane only. Magnitude determined by density.

IV. Cosmology

Isotropic, homogeneous cosmology has

$$\mathcal{R}(B) = 4\pi(\rho + P)B\cdot e_t\, e_t - \frac{1}{3}(8\pi\rho + \Lambda)B.$$

P and ρ are pressure and density, Λ is the cosmological constant, and e_t is 'rest-frame' of the universe (defined by the cosmic microwave background radiation. No other direction present.

DISPLACEMENT GAUGE AGAIN

Key quantity is

$$\mathsf{S}(a) = -\bar{\mathsf{h}}[\nabla\wedge\bar{\mathsf{h}}^{-1}(a)] = \bar{\mathsf{h}}(\dot{\nabla})\wedge\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(a).$$

But $\bar{\mathsf{h}}(a)$ picks up additional rotors under rotation gauge. Replace the directional derivatives by covariant derivatives:

$$\mathcal{S}(a) = \bar{\mathsf{h}}(\partial_b)\wedge\Big(b\cdot\dot{\nabla}\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(a) + \Omega(b)\cdot a\Big).$$

Guarantees required transformation laws

Displacements: $\mathcal{S}'(a,x) = \mathcal{S}(a,x')$

Rotations: $\mathcal{S}'(a) = R\mathcal{S}(\tilde{R}aR)\tilde{R}.$