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PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 15

SUMMARY

To find the field equations we must establish covariant forms of the field strength tensor. Dimensional analysis then tells what the correct equations are. Similar considerations apply to trajectories and enable us to identify the metric.

- The covariant field strength tensors.
- The Planck scale and the magnitude of the field strength of the displacement field.
- Further properties of the Riemann tensor.
- The second field equation.
- Trajectories.
- The metric and the link with GR.

Covariant Field Strengths

Need covariant forms of field strength. Start with rotation gauge. $\Omega(a)$ removes terms in $a\cdot\nabla R\tilde{R}$. Under displacements must have

$$\Omega(a;x) \mapsto \Omega'(a;x) = \Omega[\mathsf{f}(a);x']$$

Field strength has term in $\Omega(a)\times\Omega(b)$. Must transform to

$$\mathsf{R}(a\wedge b) \mapsto \mathsf{R}'(a\wedge b) = \mathsf{R}[\mathsf{f}(a)\wedge\mathsf{f}(b)] = \mathsf{R}[\mathsf{f}(a\wedge b);x']$$

Picks up a term in $\mathsf{f}(B)$. Remove this with suitable version of $\bar{\mathsf{h}}(a)$. Has

$$\bar{\mathsf{h}}(a) \mapsto \bar{\mathsf{h}}'(a) = \bar{\mathsf{h}}\bar{\mathsf{f}}^{-1}(a)$$

so adjoint transforms as

$$\mathsf{h}(a) \mapsto \mathsf{h}'(a) = \mathsf{f}^{-1}\mathsf{h}(a).$$

Insert this into $\mathsf{R}(B)$. Define covariant field strength

$$\mathcal{R}(B) = \mathsf{R}[\mathsf{h}(B)]$$

Factor of $\mathsf{h}(B)$ alters rotation gauge properties.

$$\bar{\mathsf{h}}(a) \mapsto \bar{\mathsf{h}}'(a) = R\bar{\mathsf{h}}(a)\tilde{R}.$$

so adjoint goes as

$$\mathsf{h}(a) \mapsto \mathsf{h}'(a) = \partial_b\langle aR\bar{\mathsf{h}}(b)\tilde{R}\rangle = \mathsf{h}(\tilde{R}aR).$$

Summarise transformation properties of $\mathcal{R}(B)$ by:

Displacements: $\mathcal{R}'(B,x) = \mathcal{R}(B,x')$

Rotations: $\mathcal{R}'(B) = R\mathcal{R}(\tilde{R}BR)\tilde{R}.$

Just what we want for a covariant tensor. Call $\mathcal{R}(B)$ the Riemann tensor. Understand rotation transformation from

$$\mathcal{R}(B) = \alpha B$$

This is 'dilate all fields by factor α'. Transformed field is

$$\mathcal{R}'(B) = R\mathcal{R}(\tilde{R}BR)\tilde{R} = R(\alpha\tilde{R}BR)\tilde{R} = \alpha B$$

Same physical information.

Displacement Gauge

Key quantity is

$$\mathsf{S}(a) = -\bar{\mathsf{h}}[\nabla\wedge\bar{\mathsf{h}}^{-1}(a)] = \bar{\mathsf{h}}(\dot{\nabla})\wedge\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(a).$$

But $\bar{\mathsf{h}}(a)$ picks up additional rotors under rotation gauge. Replace the directional derivatives by covariant derivatives:

$$\mathcal{S}(a) = \bar{\mathsf{h}}(\partial_b)\wedge\left(b\cdot\dot{\nabla}\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(a) + \Omega(b)\cdot a\right).$$

Guarantees required transformation laws

Displacements: $\mathcal{S}'(a,x) = \mathcal{S}(a,x')$

Rotations: $\mathcal{S}'(a) = R\mathcal{S}(\tilde{R}aR)\tilde{R}.$

THE FIELD EQUATIONS

Can get field equations from a Lagrangian and action principle. Will use dimensional analysis instead. Have 3 constants G, $\hbar$ and c. Fix the natural scale for interactions:

Planck length	$l_P = \sqrt{\hbar G/c^3} = 1.6 \times 10^{-35}\mathrm{m}$
Planck mass	$M_P = \hbar/L_P c = 2.2 \times 10^{-8}\mathrm{kg}$
Planck time	$t_P = L_P/c = 5.3 \times 10^{-44}\mathrm{s}.$

$\Omega(a)$ has dimensions of $(\text{length})^{-1}$ and $\bar{\mathsf{h}}(a)$ is dimensionless. So $\mathcal{S}(a)$ and $\mathcal{R}(B)$ differ in dimensions by a factor of length. Expect $\mathcal{S}(a)$ comparable in magnitude to $l_P\mathcal{R}(B)$. Extremely small! If ignore quantum effects, expect $\mathcal{S}(a)$ vanishes. Gives first field equation

$$\mathcal{S}(a) = 0.$$

(In fact, from Lagrangian analysis $\mathcal{S}(a)$ driven by quantum spin.) Does not mean $\bar{\mathsf{h}}(a)$ is a pure gauge, because $\Omega(a)$ field couples into covariant field strength! This coupling generates some dynamics. Also ensures that equations are (locally) those of GR!

Consequences

First write $\mathcal{S}(a) = 0$ in form

$$\mathcal{D}\wedge\bar{\mathsf{h}}(a) = \bar{\mathsf{h}}(\partial_b)\wedge[b\cdot\nabla\bar{\mathsf{h}}(a) + \Omega(b)\cdot\bar{\mathsf{h}}(a)] = 0$$

Version for vector field useful

$$\mathcal{D}\wedge\bar{\mathsf{h}}(A) = \bar{\mathsf{h}}(\nabla\wedge A)$$

Removes a potential ambiguity. Electromagnetic vector potential is A. Covariant version is

$$\mathcal{A} = \bar{\mathsf{h}}(A)$$

(Because A generalises $\nabla\phi$). Could generate electromagnetic field strength from A or $\mathcal{A}$. But now both give the same result

$$\mathcal{F} = \bar{\mathsf{h}}(\nabla\wedge A) = \mathcal{D}\wedge\mathcal{A} = \bar{\mathsf{h}}(F).$$

Now extend to a bivector,

$$\begin{aligned}\mathcal{D}\wedge\bar{\mathsf{h}}(a\wedge b) &= \bar{\mathsf{h}}(\partial_c)\wedge[\mathcal{D}_c\bar{\mathsf{h}}(a)\wedge\bar{\mathsf{h}}(b) + \bar{\mathsf{h}}(a)\wedge\mathcal{D}_c\bar{\mathsf{h}}(b)] \\ &= [\mathcal{D}\wedge\bar{\mathsf{h}}(a)]\wedge\bar{\mathsf{h}}(b) - \bar{\mathsf{h}}(a)\wedge[\mathcal{D}\wedge\bar{\mathsf{h}}(b)] = 0\end{aligned}$$

Result for position dependent fields is

$$\mathcal{D}\wedge\bar{\mathsf{h}}(M) = \bar{\mathsf{h}}(\nabla\wedge M)$$

Follows that

$$\mathcal{D}\wedge[\mathcal{D}\wedge\bar{\mathsf{h}}(A)] = \mathcal{D}\wedge\bar{\mathsf{h}}(\nabla\wedge A) = \bar{\mathsf{h}}(\nabla\wedge\nabla\wedge A) = 0$$

Holds for any A. Results in an algebraic identity for the Riemann tensor

$$\partial_a\wedge\mathcal{R}(a\wedge b) = 0$$

Trivector $\partial_a\wedge\mathcal{R}(a\wedge b)$ vanishes for all values of vector b. 16 constraints, so $\mathcal{R}(B)$ left with 20 degrees of freedom.

SECOND FIELD EQUATION

$\mathcal{R}(B)$ has dimensions $(\text{length})^{-2}$. Ignoring quantum effects, so only have G. Form $\mathcal{R}(B)/G$, has dimensions of energy density. Now linear in $\mathcal{R}(B)$, whereas for electromagnetism had quadratic $\mathsf{T}(a) = -\frac{1}{2}FaF$.

Expect source for gravitational fields to be matter stress-energy tensor. Need a linear function on vectors. Contract $\mathcal{R}(a\wedge b)$ and define Ricci tensor

$$\mathcal{R}(b) = \partial_a\cdot\mathcal{R}(a\wedge b)$$

(NB. Same symbol. Grade of argument distinguishes type.) Contract again to get Ricci scalar

$$\mathcal{R} = \partial_a\cdot\mathcal{R}(a)$$

Our first scalar observable. This is Lagrangian density in formal development.

To get correct equation, use Jacobi Identity. One of these for all gauge theories:

$$[D_a,[D_b,D_c]]\psi + [D_c,[D_a,D_b]]\psi + [D_b,[D_c,D_a]]\psi = 0$$

So

$$\mathcal{D}_a \mathsf{R}(b\wedge c) + \mathcal{D}_c \mathsf{R}(a\wedge b) + \mathcal{D}_b \mathsf{R}(c\wedge a) = 0$$

In electromagnetism same as $\nabla\wedge F = 0$. Covariant form in gravity more work. Result is

$$\bar{\mathsf{h}}(\partial_a)\wedge[\mathcal{D}_a\mathcal{R}(B) - \mathcal{R}(\mathcal{D}_a B)] = \dot{\mathcal{D}}\wedge\dot{\mathcal{R}}(B) = 0.$$

Called the Bianchi identity. Another useful, practical result in a simple, memorable expression. On contracting Bianchi identity, find that covariantly conserved tensor is the Einstein tensor

$$\mathcal{G}(a) = \mathcal{R}(a) - \tfrac{1}{2}a\mathcal{R}$$

Equate this with the covariant matter stress-energy tensor $\mathcal{T}(a)$,

$$\mathcal{G}(a) = \kappa\mathcal{T}(a)$$

Constant κ found from spherically symmetric solutions. Conclusion is $\kappa = 8\pi G$.

Trajectories and Tangents

Particle follows trajectory $x(\lambda)$ in the STA. Actual path has no relevance, only values of fields encountered. What relevance can be attached to the tangent vector? After displacement

$$x'(\lambda) = f(x(\lambda))$$

new tangent vector is

$$\partial_\lambda f(x(\lambda)) = \partial_\lambda x(\lambda)\cdot\nabla f(x) = \mathsf{f}(\dot{x})$$

where $\dot{x} = \partial_\lambda x(\lambda)$. Factor of $\mathsf{f}(a)$ now. Remove with suitable version of $\bar{\mathsf{h}}(a)$. Define

$$v = \mathsf{h}^{-1}(\dot{x}),$$

as the covariant tangent vector. Now transforms under rotation gauge

$$v \mapsto v' = Rv\tilde{R}$$

No change to STA trajectory. v can point wherever we like! Only constraint is v^2 invariant. Use this to distinguish spacelike, timelike and null.

Point Particle Equations of Motion

Proper (invariant) distance along trajectory now

$$s = \int_{\lambda_1}^{\lambda_2} \sqrt{|v^2|}\, d\lambda.$$

The proper time τ is therefore parameter such that

$$v^2 = 1, \quad \text{where} \quad v = \mathsf{h}^{-1}(\dot{x}) = \mathsf{h}^{-1}(\partial_\tau x).$$

No fields $\dot{v} = 0$. Make this equation covariant. Have $\partial_\tau = \dot{x}\cdot\nabla$, so use

$$\partial_\tau v + \Omega(\dot{x})\cdot v = 0$$

(Can write this more abstractly as $v\cdot\mathcal{D}\, v = 0$)

$\Omega(\dot{x})$ is a form of acceleration bivector. But have to be careful.

$\Omega(a)$ is not gauge invariant. If transform to $v' = Rv\tilde{R}$ get,

$$\begin{aligned}\partial_\tau v' + \Omega'(\dot{x})\cdot v' &= \partial_\tau v' + [R\Omega(\dot{x})\tilde{R}]\cdot v' - 2(\dot{R}\tilde{R})\cdot v' \\ &= Rv\cdot\mathcal{D}v\,\tilde{R} = 0.\end{aligned}$$

Can add 'acceleration' term $2\dot{R}\tilde{R}$ to $\Omega(a)$. field without altering physics. Can see that

$$\text{gravity} + \text{acceleration} = \text{gravity}'$$

A form of the equivalence principle. (Einstein considered lifts accelerating and at rest in a gravitational field.)

Weak Equivalence Principle

the motion of a test particle in a gravitational field is independent of its mass. Automatic — no mass in our equation. Means gravitational mass = inertial masses.

The Metric and GR

Introduce a coordinate frame. Coordinates (x^μ). Define

$$e_\mu = \partial_\mu x, \qquad e^\mu = \nabla \mathrm{x}^\mu.$$

Reciprocal to one another. Expand trajectory $x(\lambda)$in this frame

$$\partial_\lambda x = \partial_\lambda x(\mathrm{x}^\mu) = \frac{d\mathrm{x}^\mu}{d\lambda}\partial_\mu x = \frac{d\mathrm{x}^\mu}{d\lambda} e_\mu .$$

In terms of this the proper distance along a path becomes

$$\begin{aligned} s &= \int_{\lambda_1}^{\lambda_2} \sqrt{|v^2|}\, d\lambda \\ &= \int_{x_1}^{x_2} \left| \mathsf{h}^{-1}(e_\mu)\cdot \mathsf{h}^{-1}(e_\nu) d\mathrm{x}^\mu \, d\mathrm{x}^\nu \right|^{1/2} \end{aligned}$$

Comparison with GR $ds^2 = g_{\mu\nu} d\mathrm{x}^\mu \, d\mathrm{x}^\nu$ gives metric as

$$g_{\mu\nu} = \mathsf{h}^{-1}(e_\mu)\cdot \mathsf{h}^{-1}(e_\nu).$$

- In GR this is the fundamental object. Gives distance between points on a curved surface. In gauge theory it is derived.

- If $\bar{\mathsf{h}}(a)$ and $\Omega(a)$ satisfy gauge field equations then metric solves the GR Einstein equations.
- Minimise proper distance get geodesic equation. Same as $v\cdot\mathcal{D}v = 0$.
- Metric independent of rotation gauge. GR never sees this. Gauge nature of GR is hidden!
- For metric, displacements look like coordinate transformations. Confusing! NB metric cannot be transformed away by change of coordinates.

Covariant Frames

Define useful frames

$$g_\mu = \mathsf{h}^{-1}(e_\mu), \qquad g^\mu = \bar{\mathsf{h}}(\nabla x^\mu) = \bar{\mathsf{h}}(e^\mu).$$

Reciprocal because

$$g_\mu\cdot g^\nu = \mathsf{h}^{-1}(e_\mu)\cdot\bar{\mathsf{h}}(e^\mu) = e_\mu\cdot e^\nu = \delta^\nu_\mu.$$

Metric now simply

$$g_{\mu\nu} = g_\mu\cdot g_\nu.$$

Also simplify first field equation. Since

$$\mathcal{D}\wedge g^\mu = \mathcal{D}\wedge\bar{\mathsf{h}}(e^\mu) = \bar{\mathsf{h}}(\nabla\wedge e^\mu)$$

and $\nabla\wedge e^{\mu} = \nabla\wedge\nabla \mathrm{x}^{\mu} = 0$, can write $\mathcal{S}(a) = 0$ as

$$\mathcal{D}\wedge g^{\mu} = 0.$$

Now complete the link with GR. We use the abbreviation

$$\mathcal{D}_{\mu} = e_{\mu}\cdot\nabla + \Omega(e_{\mu})\times = \partial_{\mu} + \Omega(e_{\mu})\times$$

for the covariant derivative in e_{μ} direction. Act on vector g_{μ} and express result in $\{g_{\mu}\}$ frame:

$$\mathcal{D}_{\mu} g_{\nu} = \Gamma^{\lambda}_{\mu\nu} g_{\lambda}$$

Defines the Christoffel connection $\Gamma^{\lambda}_{\mu\nu}$.

Vectors transform like $\partial_{\mu} x$ (with $\mathsf{f}(a)$) or like ∇ (with $\bar{\mathsf{f}}^{-1}(a)$). Mathematicians call these vectors and 1-forms. Like to keep them separate. Identified by the metric. We use $\bar{\mathsf{h}}$-field to make all same type — covariant vectors. Then just have rotor group transformations.