March 8, 1999

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 16

SUMMARY

The solution for the gravitational fields outside a spherically symmetric source can be motivated by simple Newtonian considerations. In this final lecture we discuss some of the properties of this solution.

- The vacuum equations and properties of the vacuum Riemann tensor.
- Spherically-symmetric fields.
- Observers in free-fall.
- Incoming and outgoing photons.
- The horizon.
- Stationary observers.
- The Schwarzschild metric.

The Vacuum Equations

On dimensional grounds have argued that

$$\mathcal{G}(a) = \mathcal{R}(a) - \tfrac{1}{2}a\mathcal{R} = 8\pi G\mathcal{T}(a)$$

where $\mathcal{T}(a)$ is matter stress-energy tensor. For vacuum fields outside source have

$$\mathcal{G}(a) = 0$$

Contracting with ∂_a get

$$\partial_a \cdot \mathcal{R}(a) - \tfrac{1}{2}\partial_a \cdot a\,\mathcal{R} = -\mathcal{R} = 0$$

so vacuum equations equivalent to

$$\partial_a \cdot \mathcal{R}(a\wedge b) = \mathcal{R}(a) = 0$$

Combine with symmetry relation $\partial_a \wedge \mathcal{R}(a\wedge b)$ see that vacuum Riemann satisfies

$$\partial_a \mathcal{R}(a\wedge b) = 0$$

A set of four equations. Set $b = \gamma_0$, get

$$\partial_a \mathcal{R}(a\wedge\gamma_0) = \gamma^i \mathcal{R}(\gamma_i\gamma_0) = 0$$

$\times$ by γ_0, get

$$\boldsymbol{\sigma}_1 \mathcal{R}(\boldsymbol{\sigma}_1) + \boldsymbol{\sigma}_2 \mathcal{R}(\boldsymbol{\sigma}_2) + \boldsymbol{\sigma}_3 \mathcal{R}(\boldsymbol{\sigma}_3) = 0$$

Three more equations like this (exercise). Conclusion is that

$$\mathcal{R}(IB) = I\mathcal{R}(B)$$

A remarkable relation, true for all vacuum fields.

- Relativists call this property duality.
- Means that vacuum fields have a natural complex structure. Can view as a mapping between 3-d complex spaces.
- Map is symmetric and traceless so 5 complex components, 10 real dof.
- 6 of these are gauge, so 4 physical observables (the 2 complex eigenvalues).

Gravity not looking so complicated now!

Spherically Symmetric Sources

Newtonian equation for a point particle is

$$\ddot{r} = -\frac{GM}{r^2},$$

which integrates to give

$$\frac{1}{2}\dot{r}^2 = \frac{GM}{r} + \text{constant.}$$

This must survive in some form! Introduce polar coordinates

$$t = x\cdot\gamma_0 \qquad \cos\theta = x\cdot\gamma^3/r$$
$$r = \sqrt{(x\wedge\gamma_0)^2} \qquad \tan\phi = (x\cdot\gamma^2)/(x\cdot\gamma^1).$$

The associated coordinate frame is

$$e_t = \gamma_0$$
$$e_r = x\wedge\gamma_0\,\gamma_0/r = \sin\theta(\cos\phi\,\gamma_1 + \sin\phi\,\gamma_2) + \cos\theta\,\gamma_3$$
$$e_\theta = r\cos\theta(\cos\phi\,\gamma_1 + \sin\phi\,\gamma_2) - r\sin\theta\,\gamma_3$$
$$e_\phi = r\sin\theta(-\sin\phi\,\gamma_1 + \cos\phi\,\gamma_2).$$

Assume source at rest in γ_0 frame. t is coordinate time and is a gauge-dependent concept, whereas in $\dot{r}$ derivative must be proper time.

But must fix gauge to write down solution. So choose t equal to proper time for freely-falling observers. Must agree on time, so set all at rest at infinity. Gives

$$\dot{r} = -\sqrt{(2GM/r)}$$

and the paths have

$$\dot{x} = \frac{dx}{dt} = e_t - \sqrt{(2GM/r)}e_r.$$

Covariant vector is $v = \mathsf{h}^{-1}(\dot{x})$, $v^2 = 1$. Again, free to choose this vector. Keep physics simple and set $v = e_t$. (Another gauge choice). Now

$$e_t = v = \mathsf{h}^{-1}(\dot{x})$$

so

$$\mathsf{h}(e_t) = \dot{x} = e_t - \sqrt{(}2GM/r)e_r.$$

This gives us a plausible term in the $\bar{\mathsf{h}}$-field. Make bold ansatz that this is only term! That is

$$\mathsf{h}(a) = a - \sqrt{(}2GM/r)\, a\cdot e_t\, e_r,$$

with adjoint

$$\bar{\mathsf{h}}(a) = a - \sqrt{(}2GM/r)\, a\cdot e_r\, e_t.$$

This works! The equation $\mathcal{D}\wedge g^\mu = 0$ specifies the $\Omega(a)$:

$$\Omega(e_t) = \frac{GM}{r^2} e_r e_t \qquad \Omega(e_r) = -\frac{GM}{ur^2} e_r e_t$$
$$\Omega(e_\theta) = u/r\, e_\theta e_t \qquad \Omega(e_\phi) = u/r\, e_\phi e_t$$

where

$$u = -\sqrt{(}2GM/r)$$

Write $\boldsymbol{\sigma}_r = e_r e_t$ for relative radial vector. See that $\Omega(e_t)$ term ('acceleration in e_t direction') is $GM/r^2\, \boldsymbol{\sigma}_r$.
Follow through to Riemann tensor. End result is simply

$$\mathcal{R}(B) = \frac{-M}{2r^3}(B + 3\boldsymbol{\sigma}_r B \boldsymbol{\sigma}_r)$$

Presence of r here makes it observable (via tidal force). Now both t and r are physically-defined. Not just arbitrary coordinates.

Proof

Need to establish that we have a vacuum solution. Have

$$\partial_a a\wedge b = \partial_a(ab - a\cdot b) = (n-1)b = 3b$$

since we are working in 4-d. Also

$$\partial_a\, a\cdot(b\wedge c) = \partial_a(a\cdot b\, c - a\cdot c\, b) = bc - cb = 2b\wedge c,$$

So again proved that

$$\partial_a a\cdot B = 2B \quad \forall B$$

It follows that in 4-d

$$\begin{aligned}\partial_a Ba &= \partial_a(Ba - aB) + \partial_a aB \\ &= -2\partial_a a\cdot B + 4B = -4B + 4B = 0\end{aligned}$$

Can now prove have a vacuum solution (and simultaneously verify symmetry properties)

$$\begin{aligned}\partial_a(a\wedge b + 3\boldsymbol{\sigma}_r\, a\wedge b\, \boldsymbol{\sigma}_r) &= 3b + 3\partial_a \boldsymbol{\sigma}_r (ab - a\cdot b)\boldsymbol{\sigma}_r \\ &= 3b - 3b\boldsymbol{\sigma}_r\boldsymbol{\sigma}_r = 0\end{aligned}$$

This is much quicker than tensor calculus, since then have no option but to work out each term in $\mathcal{R}(a)$.

Turns out that this solution is unique. Fields outside a spherically symmetric source all gauge equivalent to solution here.

Freely-Falling Observers

Motivated our solution from Newtonian paths. But know that free-fall is actually determined by

$$v\cdot\mathcal{D}v = \dot{v} + \Omega(\dot{x})\cdot v = 0$$

So is $v = e_t$ a solution now? Still have

$$\dot{x} = \mathsf{h}(v) = \mathsf{h}(e_t) = e_t + ue_r$$

So

$$\Omega[\mathsf{h}(e_t)] = \Omega(e_t) + u\Omega(e_r) = 0$$

and it follows that

$$e_t\cdot\mathcal{D}e_t = \partial_\tau e_t + \Omega[\mathsf{h}(e_t)]\cdot e_t = 0$$

So observers freely-falling from infinity do follow the Newtonian trajectory. Justifies the whole approach.

Now consider arbitrary, radial free-fall. Must have

$$v = \mathrm{e}^{\alpha\boldsymbol{\sigma}_r}\, e_t = \cosh(\alpha)e_t + \sinh(\alpha)e_r$$

so

$$\dot{v} = \dot{\alpha}\boldsymbol{\sigma}_r\cdot v$$

and

$$
\begin{aligned}
\Omega(\dot{x}) &= \cosh(\alpha)\Omega[\mathsf{h}(e_t)] + \sinh(\alpha)\Omega[\mathsf{h}(e_r)] \\
&= -\sinh(\alpha)\frac{GM}{r^2 u}\boldsymbol{\sigma}_r
\end{aligned}
$$

The free-fall equation reduces to

$$\dot{\alpha} = \sinh(\alpha)\frac{GM}{r^2 u}$$

and $\dot{x} = \mathsf{h}(v)$ gives

$$
\begin{aligned}
\dot{t} &= \cosh(\alpha) \\
\dot{r} &= \sinh(\alpha) + u\cosh(\alpha)
\end{aligned}
$$

Enough to plot trajectories. Taking second derivative of $\dot{r}$ equation, get

$$\ddot{r} = -\frac{GM}{r^2}$$

so Newton still present! Now an equation in terms of local observables. Also see that

$$\dot{r}/\cosh(\alpha) = \tanh(\alpha) - \surd(2GM/r)$$

so that once $2GM/r > 1$, have $\dot{r}$ necessarily negative! No way to escape. This corresponds to the escape velocity $\sqrt{2GM/r}$ being greater than the speed of light. Still all Newtonian.

Photon Paths

Use geometric optics approximation (photons as point particles). Follow null trajectories,

$$k = \mathsf{h}^{-1}(\dot{x}), \qquad k^2 = 0$$

Still have $k \cdot \mathcal{D}k = 0$. For radial infall

$$k = \omega(e_t - e_r)$$

ω is frequency measured by free-falling observers (from infinity). Get path

$$\dot{x} = \mathsf{h}(v) = \omega[e_t - (1 + \surd(2GM/r))e_r]$$

so

$$\frac{dr}{dt} = -(1 + \surd(2GM/r))$$

Integrate to get path. For equation of motion need

$$\Omega(\dot{x}) = \omega\Omega[\mathsf{h}(e_t)] - \omega\Omega[\mathsf{h}(e_r)] = \omega\frac{GM}{r^2 u}\boldsymbol{\sigma}_r$$

Get

$$\dot{\omega} = \omega^2\frac{GM}{r^2 u}$$

More usefully expressed in terms of r. Use

$$\dot{r} = -\omega[1 + \surd(2GM/r)]$$

to get

$$\frac{1}{\omega}\frac{d\omega}{dr} = \frac{GM}{r}\frac{1}{2GM + \sqrt{(2GMr)}} = \frac{1}{2r}\frac{1}{\sqrt{r/r_S} + 1},$$

where $r_S = 2GM$ is Schwarzschild radius. Tells us how frequency measured by free-falling observers changes with radius. Nothing problematic until $r = 0$.

Outgoing Photons

Repeat previous for outgoing photons. Now have

$$k = \omega(e_t + e_r)$$

and path is

$$\dot{x} = \mathsf{h}(v) = \omega[e_t + (1 - \sqrt{(2GM/r)})e_r]$$

so

$$\frac{dr}{dt} = 1 - \sqrt{(2GM/r)}$$

But now, when $r < 2GM$ path is still inwards.

- Inside $r = 2GM$, not even light can escape — Called the event horizon.
- Object collapsed to within its event horizon, must carry on collapsing to form central singularity.
- Remaining object called a black hole.

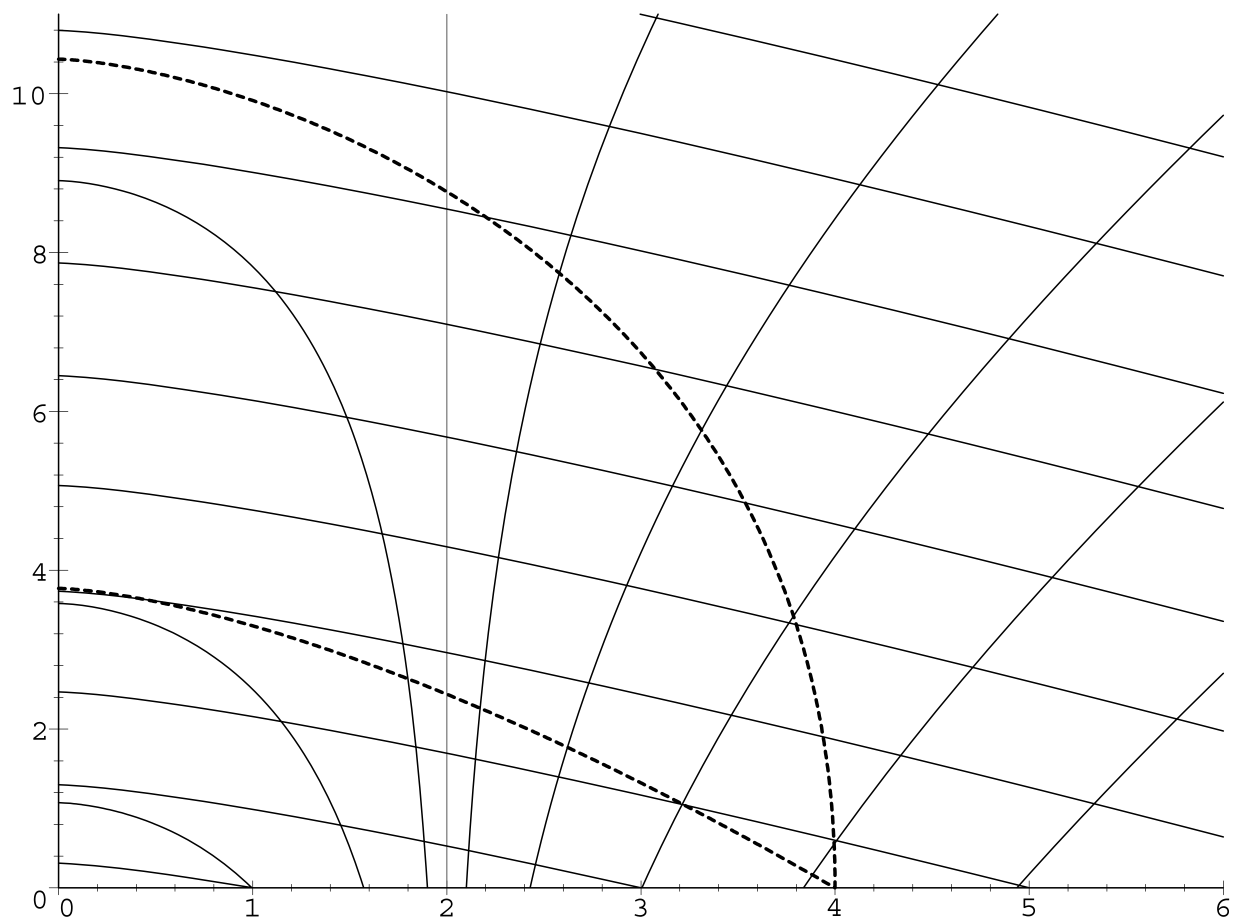
All paths terminate on singularity. Problematic for GR. Less so for gauge theory — not much worst than Coulomb singularity. Treated using integral equations.

Also find now that

$$\frac{1}{\omega}\frac{d\omega}{dr} = \frac{GM}{r}\frac{1}{2GM - \sqrt{(2GMr)}} = -\frac{1}{2r}\frac{1}{\sqrt{r/r_S} - 1},$$

Negative outside r_S. Photons red-shifted as they climb out of gravitational well.

Summary



- Solid lines are photon paths. Horizon at $r = 2$.
- Broken lines are matter free-fall. I released from rest at $r = 4$. II from infinity.
- Photons near horizon are redshifted and take a long time to escape.

As seen from external observers, any object falling through the horizon appears to hover there and get redshifted out of existence.

STATIONARY OBSERVERS

As well as free-fall, look at physics from point of view of stationary observers. Have constant r, θ, ϕ, so

$$\dot{x} = \dot{t} e_t$$

Follows that

$$v = \dot{t}(e_t + \sqrt{(2GM/r)} e_r)$$

But need $v^2 = 1$ for proper time, so

$$\dot{t}^2(1 - 2GM/r) = 1, \qquad \dot{t} = (1 - 2GM/r)^{-1/2}$$

which is constant. See that it is only possible to remain at rest outside the horizon. (Fairly obvious, but two concepts not equivalent if black hole rotates).

Define covariant acceleration bivector by

$$v\cdot\mathcal{D}v\,v = \dot{v}v + \Omega(\dot{x})\cdot v\,v$$

Gives force needed to get specified path. For stationary observers

$$v\cdot\mathcal{D}v\,v = \Omega(\dot{x}) = \frac{GM}{r^2(1-2GM/r)^{1/2}}\boldsymbol{\sigma}_r$$

So mass m needs to apply force

$$\frac{GMm}{r^2} \times (1-2GM/r)^{-1/2}$$

Gets large close to the horizon. Can now look at physics from point of view of these accelerating observers.

Example

Second observer velocity γ_0. When coincident, get relative velocity

$$\frac{v\wedge\gamma_0}{v\cdot\gamma_0} = \sqrt{(2GM/r)}\boldsymbol{\sigma}_r$$

No different to Newtonian result! And gauge invariant

Schwarzschild Metric

Finish by looking at the link with GR. Our h-field produces line element

$$
\begin{aligned}
ds^2 =& (1-2GM/r)dt^2 - 2\sqrt{(2GM/r)}dt\,dr \\
& - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)
\end{aligned}
$$

Off-diagonal term makes this look more complicated than our simple h-field. In GR remove the off-diagonal term by introducing a new time coordinate t' such that

$$dt = dt' + \alpha(r)dr$$

so that recover the Schwarzschild form

$$
\begin{aligned}
ds^2 =& (1-2GM/r)dt'^2 - (1-2GM/r)^{-1}dr^2 \\
& - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)
\end{aligned}
$$

But setting $r = 2GM$ in first form, see that coefficient of cross term is -2.
Cannot change this unless $\alpha(1-2GM/r)$ is finite at the horizon, and transformation is singular there. In gauge theory terms the displacement is not defined globally, so if a horizon is present, the gauge is not valid

GR expresses this differently (coordinates only valid locally). In some situations this can produce differences. Outside horizon, get complete agreement.